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Unknown properties of bimodal power Laplace distribution

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Abstract. The main goal of this article is to present the unknown properties of the bimodal power Laplace (BPL) distribution such as: inflection points, Moors' measure, pseudorandom number generator, entropy and Fisher information matrix. The secondary goals involve: a chronological overview of the large family of Laplace distributions, a brief review of the known properties of the analysed distribution and an estimation of the unknown parameters of the BPL distribution.

The BPL distribution has been defined by means of the standard Laplace distribution. The research has shown that the probability density function of the BPL distribution can also be obtained by applying the Fernández-Steel transformation to the Weibull distribution with a scale parameter equal to 1. Real data examples demonstrate that the BPL distribution is a flexible model that proves to be a valuable addition to the existing distributions for modelling bimodal data.

Keywords: Laplace distribution, double Weibull distribution, modelling bimodal data
JEL: C02, C10, C13, C15, C50

Nieznane właściwości dwumodalnego rozkładu potęgowego Laplace'a

Streszczenie. Głównym celem artykułu jest przedstawienie nieznanych właściwości dwumodalnego rozkładu potęgowego Laplace'a (BPL), takich jak: punkty przegięcia, miara Moorsa, generator liczb pseudolosowych, entropia i macierz informacji Fishera. Celami pobocznymi są: chronologiczny przegląd dużej rodziny rozkładów Laplace'a, krótka prezentacja znanych właściwości analizowanego rozkładu oraz estymacja nieznanych parametrów rozkładu BPL.

Rozkład BPL zdefiniowano z wykorzystaniem standardowego rozkładu Laplace'a i stwierdzono, że funkcję gęstości prawdopodobieństwa rozkładu BPL można także uzyskać, stosując transformację Fernándeza-Steela do rozkładu Weibulla z parametrem skali równym 1. Przykłady danych rzeczywistych pokazują, że rozkład BPL jest elastycznym modelem, który zasługuje na dodanie go do istniejących rozkładów modelowania danych dwumodalnych.

Słowa kluczowe: rozkład Laplace'a, podwójny rozkład Weibulla, modelowanie danych dwumodalnych

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1. Introduction

The Laplace (L) distribution was defined by Laplace (1986). $L(a \in R, b > 0)$ with the probability density function (PDF)

$$f_L(x; a, b) = \frac{1}{2b} \exp\left(-\left|\frac{x-a}{b}\right|\right) \quad (x \in R),$$

also called the double exponential distribution, and its variants are becoming popular in many areas of science and engineering (Cordeiro & Lemonte, 2011). This distribution is often used for modelling phenomena with 'heavier than normal tails' (see e.g. Bagchi et al., 1983; Chen, 2002; Dadi & Marks, 1987; Damsleth & El-Shaarawi, 1989; Easterling, 1978; Hoaglin et al., 1983; Hsu, 1979; Johnson et al., 1995; Manly, 1976; Puig & Stephens, 2000).

The unimodal distributions are frequently used in theoretical statistical studies. However, in applied statistics, there are many situations in which the unimodal distributions cannot be fitted to the data. For example, the distribution of data outside the control zone in quality control or outlier observations in linear models and time series may have to be bimodal. These situations occur when the probability of the recorded data is proportional to the increasing function of the absolute value of the deviations (Alavi, 2012).

Data demonstrating bimodal behaviour is produced in many different disciplines. One of the most relevant phenomena that can be explained through bimodal distributions are disease patterns. By understanding the reason behind the multimodality of some cancer incidence curves, practitioners may enhance their knowledge of the cancer developmental process and help distinguish the potential features that identify cancer and separate a specific type of cancer from all other types (Gómez-Déniz et al., 2021). Urinary mercury excretion also showed two peaks (see e.g. Ely et al., 1999). In the material characteristics, studied by Dierickx et al. (2000), the data on grain size distribution revealed a bimodal structure. In meteorology, Zhang et al. (2003) indicated that water vapor in the tropics tends to have a bimodal distribution. The occurrence of bimodality has also implications in bio-geosciences (see e.g. Hirota et al., 2011).

Other articles devoted to bimodal distributions include those by Bolfarine et al. (2018), Du (2015), Eugene et al. (2002), Harandi and Alamatsaz (2013), Hassan and El-Bassiouni (2016), Hassan and Hijazi (2010), Vila et al. (2021) and Zabolotnii et al. (2018).

Harandi and Alamatsaz (2013) defined the bimodal Laplace (BL) distribution. The PDF of $BL(a \in R, b > 0)$ is

$$f_{BL}(x; a, b) = \frac{1}{4b}\left(\frac{x-a}{b}\right)^2 \exp\left(-\left|\frac{x-a}{b}\right|\right) \quad (x \in R).$$

Zabolotnii et al. (2018) used the Laplacian (bimodal) peak distribution (LP). The PDF of $LP(a \in R, b > 0)$ is given by

$$f_{LP}(x; a, b) = \frac{1}{4b}\left[\exp\left(-\left|\frac{x-a}{b}\right|\right) + \exp\left(-\left|\frac{x+a}{b}\right|\right)\right] \quad (x \in R).$$

The article focuses on defining the bimodal power Laplace (BPL) distribution by means of the standard Laplace distribution. The main goal of this article is to present the unknown properties of the BPL distribution, such as: inflection points, Moors' measure, pseudorandom number generator, entropy and Fisher information matrix. The secondary goals involve: a chronological overview of the large family of Laplace distributions, a brief review of the known properties of the analysed distribution and an estimation of the unknown parameters of the BPL distribution. A simulation study was carried out to assess the properties of the parameter estimators. Finally, illustrative examples are presented. The main R codes and proofs of some theorems are provided in the Appendix.

There are many asymmetric extensions to generalising the Laplace distribution. Basically, all these existing asymmetric Laplace distributions are either a percent mixture of the double exponential distribution or a part of it. These distributions in chronological order include: the skew-log-Laplace (Hartley & Revankar, 1974), asymmetric-Laplace (Hinkley & Revankar, 1977), log-Laplace (Inoue, 1979), skew-Laplace (Fieller et al., 1992), asymmetric-Laplace (Koenker & Machado, 1999), asymmetric-Laplace (Kotz et al., 2001), called the generalized asymmetric-Laplace distribution by Yu and Zhang (2005), asymmetric-Laplace (Kotz et al., 2002), normal-Laplace (Reed & Jorgensen, 2004), skew-Laplace (Aryal & Nadarajah, 2005), skew-Laplace-Laplace (Ali et al., 2009), beta-Laplace (Cordeiro & Lemonte, 2011), Esscher-transformed-Laplace (Sebastian & Dais, 2012), skew-Laplace-normal and generalized skew-symmetric-Laplace-normal (Nekoukhou & Alamatsaz, 2012), alpha-skew-Laplace (Harandi & Alamatsaz, 2013), Kumaraswamy-Laplace (Nassar, 2016), transmuted Laplace distribution (Hady & Shalaby, 2016), based on the quadratic rank transmutation map studied by Shaw and Buckley (2009), flexible-skew-Laplace (Yilmaz, 2016), Laplace-exponential (Kiprotich, 2018), Balakrishnan-alpha-skew-Laplace (Shah et al., 2019), reduced beta-skewed-Laplace (Arowolo et al., 2019), generalized-transmuted-Laplace (Radwan, 2020), lpha-beta-skew-Laplace (Shah & Hazarika, 2020), generalized skew-log-Laplace (Khandeparkar & Dixit, 2021), beta-skew-Laplace, truncated beta-skew-Laplace, exponentiated beta-skew-Laplace and exponentiated Laplace (Tovar-Falón & Martínez-Flórez, 2022), Balakrishnan-alpha-beta-skew-Laplace (Shah et al., 2023) and Kumara-swamy-Esscher-transformed-Laplace (George & Rimsha, 2023) distribution.

2. Properties of the bimodal power Laplace (or double Weibull) distribution

2.1. PDF: how the distribution can be created

The PDFs of the BPL distribution can be obtained by applying two methods. The first one is as follows. Let $U \sim L(0,1)$ and let us define a random variable X as

$$X = \left[a - b(-U)^{\frac{1}{c}}\right] I(U < 0) + \left[a + bU^{\frac{1}{c}}\right] I(U \geq 0), \tag{1}$$

where $a \in R, b > 0$ are the position and scale parameters, respectively. As a result of simple transformations, for $U \geq 0$, we receive

$$U = \left(\frac{X-a}{b}\right)^c \Rightarrow \frac{dU}{dX} = \frac{c}{b}\left(\frac{X-a}{b}\right)^{c-1},$$

$$f(x; a, b, c) = \frac{c}{2b}\left(\frac{x-a}{b}\right)^{c-1} \exp\left(-\left|\frac{x-a}{b}\right|^c\right) \ (x \geq a).$$

For $U < 0$, we have

$$U = -\left(\frac{a-X}{b}\right)^c \Rightarrow \frac{dU}{dX} = \frac{c}{b}\left(\frac{a-X}{b}\right)^{c-1},$$

$$f(x; a, b, c) = \frac{c}{2b}\left(\frac{a-x}{b}\right)^{c-1} \exp\left(-\left|\frac{a-x}{b}\right|^c\right) \ (x < a).$$

Finally, the PDF of the BPL distribution has the following form:

$$f_{BPL}(x; a, b, c) = \frac{c}{2b}\left|\frac{x-a}{b}\right|^{c-1} \exp\left[-\left|\frac{x-a}{b}\right|^c\right] \ (x \in R; a \in R; b, c > 0), \tag{2}$$

The second method is that proposed by Fernández and Steel (1998). They defined a class of distributions with skewness parameter $d > 0$ in the form of

$$g(x; d) = \frac{d}{1+d^2}\left[f(xd) I(x \geq 0) + f\left(\frac{-x}{d}\right) I(x < 0)\right], \tag{3}$$

where a continuous distribution on $(0,\infty)$ with base density f is transformed into the whole real line. The special case for $d = 1$ results in a symmetric distribution, i.e.

$$g_s(x) = \frac{1}{2} f(|x|). \tag{4}$$

The PDF of the Weibull (W) distribution with the scale parameter equal to 1 and shape parameter $c > 0$ is given by

$$f_W(x;c) = cx^{c-1}\exp[-x^c]\ (x > 0). \tag{5}$$

By applying transformation (4) to the Weibull distribution (5), we obtain a distribution with a

$$f_{DW}(x;c) = \frac{c}{2}|x|^{c-1}\exp(-|x|^c)\ (x \in R) \tag{6}$$

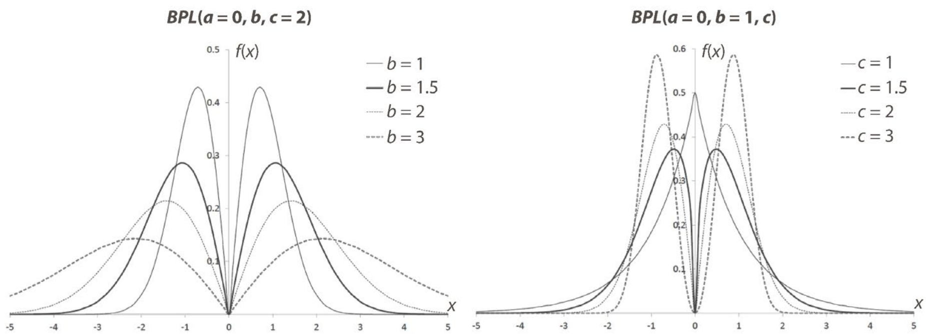
PDF, called the double Weibull (DW) distribution devised by Balakrishnan and Kocherlakota (1985).

Balakrishnan and Kocherlakota defined the DW distribution in the context of order statistics of its estimation such as the expected values, variances and covariances of the order statistics tabled for estimations on $n \in [2,10]$. Rao and Narasimham (1989) extended the results for samples $n \in [11,20]$. Jurić (2019) and Jurić and Kozubowski (2004, 2005) also studied some properties of the DW distribution. It should be noted, however, that the DW distribution introduced by Perveen et al. (2017) is defined completely differently.

The PDFs of the $BPL(0,1,c)$ distribution (2) and $DW(c)$ distribution (6) are the same and therefore only the name BPL distribution will be used in the rest of the article, as the work is focused on the family of the Laplace distributions.

Figure 1 plots the PDF of the $BPL(a,b,c)$ for selected values of parameters. The PDF of this distribution is symmetrical. If $c = 1$, the $BPL(a,b,c = 1)$ is the $L(a,b)$. If $c > 1$, the PDF has two modes with the same height. If $c \in (0,1]$ then the BPL is a unimodal distribution with the mode at 0.

Figure 1. The PDF of the $BPL(a, b, c)$ for selected values of parameters



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2.2. Cumulative distribution function

Theorem 1. Let $X \sim BPL(a, b, \mathrm{c})$, then the cumulative distribution function (CDF) of X is given by (see also Jurić, 2019)

$$(x; a, b, c) = \begin{cases} \frac{1}{2} \exp\left[-\left(\frac{a-x}{b}\right)^c\right] \ (x < a) \\ 1 - \frac{1}{2} \exp\left[-\left(\frac{x-a}{b}\right)^c\right] \ (x \geq a) \end{cases} \tag{7}$$

or

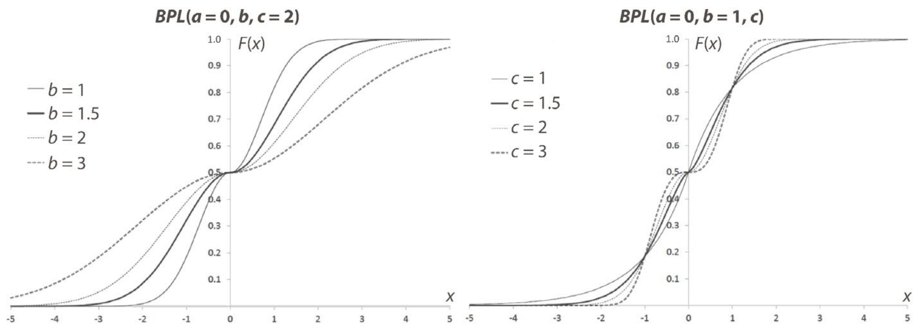
$$F(x; a, b, c) = \frac{1}{2} + \frac{1}{2} \operatorname{sgn}\left[\left(\frac{x-a}{b}\right)^c\right]\left[1 - \exp\left(-\left|\frac{x-a}{b}\right|^c\right)\right],$$

where $\operatorname{sgn}(x) = 1 \cdot I(x > 0) + 0 \cdot I(x = 0) + (-1) \cdot I(x < 0)$.

Proof. Formula (7) can be easily obtained based on (2) using substitutions $(a - x)^c b^{-c} = u$ for $x < a$ and $(x - a)^c b^{-c} = u$ for $x \geq a$ when integrating.

Figure 2 plots the CDF of the $BPL(a, b, c)$ for selected values of parameters. For $c = 1$, we obtain the CDF of the L distribution. For $c > 1$, we obtain two CDFs of the LD placed in tiers. The first sub-CDF is between 0 and 0.5, while the other one is between 0.5 and 1.

Figure 2. The CDF of the $BPL(a, b, c)$ for selected values of parameters



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2.3. Modal values, inflection points and quantiles

Theorem 2. Let $X \sim BPL(a, b, c), c \geq 1$. The modal values of $f(x; a, b, c)$ are equal to (see also Jurić, 2019)

$$x_{m1} = a - b\left(1 - \frac{1}{c}\right)^{1/c}, \quad x_{m2} = a + b\left(1 - \frac{1}{c}\right)^{1/c}. \tag{8}$$

If $c = 1$, then $x_{m1} = x_{m2} = a$. $f(x; a, b, c), c > 1$ monotonically increases on interval $(-\infty, x_{m1}) \vee (a, x_{m2})$ and monotonically decreases on interval $(x_{m1}, a) \vee (x_{m2}, \infty)$. The proof of (8) is presented in Appendix A.

Theorem 3. Let $X \sim BPL(a, b, c)$. The inflection points of $f(x; a, b, c)$ are given by the following formulas:

$$\begin{aligned} x_{\mathrm{I,II}} &= a - b\left(\frac{3c \mp \sqrt{(c-1)(5c-1)} - 3}{2c}\right)^{\frac{1}{c}}, \\ x_{\mathrm{III,IV}} &= a + b\left(\frac{3c \mp \sqrt{(c-1)(5c-1)} - 3}{2c}\right)^{\frac{1}{c}}, \end{aligned} \tag{9}$$

where $c > 2$. The proof of (9) is presented in Appendix B.

Inflection points are used in data analysis, where we observe changes in the rate of growth or decline of the studied phenomenon. In economics, inflection points can represent turning points in economic processes. They can also be used to analise the behavior of consumers and businesses.

Theorem 4. Let $X \sim BPL(a, b, c)$. The p-th $(0 < p < 1)$ quantile is given by (see also Jurić, 2019)

$$x_p = \left(a - b[-\ln(2p)]^{1/c}\right) I(p < 0.5) + \left(a + b[-\ln(2 - 2p)]^{1/c}\right) I(p \geq 0.5). \quad (10)$$

From (10), it follows that $x_{0.5} = a$. The proof of (10) based on (7) is trivial.

2.4. Moments and Moors' measure

Variance, skewness and kurtosis do not depend on the position parameter and therefore the non-central moments of the $BPL(a, b, c)$ are calculated for $a = 0$.

Theorem 5. Let $\tilde{X} \sim BPL(0, b, c)$. The k-th $(k \in Z)$ central μ_k and non-central α_k moments are given by (see also Jurić, 2019)

$$\mu_k = \alpha_k = E\left(\tilde{X}^k\right) = 0.5\Gamma\left(\frac{k}{c} + 1\right)[(-b)^k + b^k], \quad (11)$$

where $\Gamma(\cdot)$ is Euler's gamma function.

The proof of (11) is presented in Appendix C.

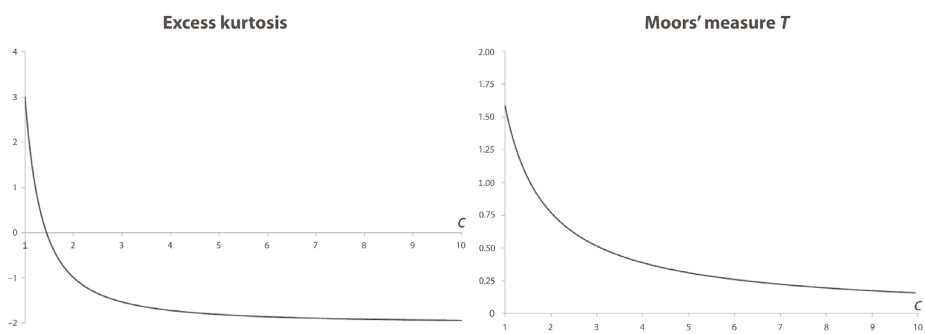
Theorem 6. Let $X \sim BPL(a, b, c)$. The excess kurtosis of X is given by

$$\bar{\gamma}_2(X) = c\Gamma\left(\frac{4}{c}\right)\left[\Gamma\left(\frac{2}{c}\right)\right]^{-2} - 3. \quad (12)$$

The proof of (12), based on non-central moments, is easy to carry out.

Figure 3 plots the excess kurtosis of the $BPL(a, b, c)$ as a function of c. Excess kurtosis $\bar{\gamma}_2(c)$ decreases as shape parameter c increases. If $c \to \infty$, then $\bar{\gamma}_2(c) \to -2$ and $\bar{\gamma}_2(1) = 3$. The excess kurtosis is positive for $c \in [1, 1.442)$, equals 0 for $c = 1.442$ and is negative for $c > 1.442$.

Figure 3. The excess kurtosis of the *BPL*(*a*, *b*, *c*) as a function of *c* and Moors' measure *T* as a function of *c*



Source: authors' work.

Moors (1988) proposed a measure of distribution flattening based on quantiles in the form of

$$T = \frac{x_{\frac{7}{8}} - x_{\frac{5}{8}} + x_{\frac{3}{8}} - x_{\frac{1}{8}}}{x_{\frac{3}{4}} - x_{\frac{1}{4}}}. \tag{13}$$

Measure (13) is a quantile alternative for kurtosis and exists even for distributions for which no moments exist. Of course, this measure does not depend on the position and scale parameters.

Theorem 7. Let $X \sim BPL(a, b, c)$. Moors' measure of X is given by

$$T(c) = 2^{1/c} - \left[2 - \frac{\ln(3)}{\ln(2)}\right]^{\frac{1}{c}}. \tag{14}$$

The proof of (14), based on formulas (10) and (13), is trivial.

Figure 3 (right) shows measure T as a function of shape parameter c. The $T(c)$ is a strictly decreasing function for the initial c values. If $c \to \infty$, then $T(c) \to 0$.

2.5. Pseudo-random number generator

Theorem 8. Let $X \sim BPL(a, b, c)$, $U \sim L(0,1)$, $R \sim Unif(0, 1)$. The pseudo-random number generator of X is given by the following formulas:

$$X = \left\{a - b(-U)^{\frac{1}{c}}\right\} I(U < 0\,) + \left\{a + b(-U)^{\frac{1}{c}}\right\} I(U \geq 0\,), \tag{15}$$

$$X = \{a - b[-\ln(2R)]^{1/c}\} I(p < 0.5) + \{a + b[-\ln(2 - 2R)]^{1/c}\} I(p \geq 0.5). \tag{16}$$

The proof of (15) follows from (5). The proof of (16), using inversion of the CDF (7), is easy to carry out.

2.6. Shannon, Tsallis and Renyi entropies

In information theory, the entropy of a random variable quantifies the average level of uncertainty or information associated with the variable's potential states or possible outcomes. This measures the expected amount of information needed to describe the state of the variable, considering the distribution of probabilities across all potential states (Beale, 1996).

Theorem 9. Let $f(x;a,b,c)$ be PDF (2). Shannon entropy S, Renyi entropy of order α, denoted by R_α, and Tsallis entropy of order α, denoted by T_α of the $BPL(a,b,c)$, are given as

$$S(b,c) = \ln\left(\frac{2b}{c}\right) + \frac{c-1}{c}\gamma + \Gamma\left(\frac{1}{c}+1\right), \tag{17}$$

$$T_\alpha(a,b,c) = \frac{c^\alpha}{(\alpha-1)(2b)^\alpha} Int(\alpha,a,b,c) - 1, \tag{18}$$

$$R_\alpha(a,b,c) = \frac{\alpha}{(1-\alpha)}\ln\left(\frac{c}{2b}\right) + \frac{1}{(1-\alpha)}\ln[Int(\alpha,a,b,c)], \tag{19}$$

respectively, where

$$Int(\alpha,a,b,c) = \int_{-\infty}^{\infty} \left|\frac{x-a}{b}\right|^{\alpha(c-1)} \exp\left(-\left|\frac{x-a}{b}\right|^c\right)^\alpha dx \tag{20}$$

and $\gamma \approx 0.577$ is the Euler-Mascheroni constant.

Proof. The Shannon (S) entropy is defined as (Shannon, 1948; Tabass et al., 2016)

$$S(a,b,c) = -\int_{-\infty}^{\infty} f(x;a,b,c)\ln[f(x;a,b,c)]dx,$$

then

$$S(a,p) = I_1 + I_2 + I_3, \tag{21}$$

where

$$I_1 = \frac{-c}{2b}\ln\left(\frac{c}{2b}\right)\int_{-\infty}^{\infty}\left|\frac{x-a}{b}\right|^{c-1}\exp\left(-\left|\frac{x-a}{b}\right|^c\right)dx = -\ln\left(\frac{c}{2b}\right), \tag{22}$$

$$I_2 = \frac{-c(c-1)}{2b}\int_{-\infty}^{\infty}\left|\frac{x-a}{b}\right|^{c-1}\exp\left(-\left|\frac{x-a}{b}\right|^c\right)\ln\left|\frac{x-a}{b}\right| dx = I_4 + I_5, \tag{23}$$

$$I_3 = \frac{c}{2b}\int_{-\infty}^{\infty}\left|\frac{x-a}{b}\right|^{c}\exp\left(-\left|\frac{x-a}{b}\right|^c\right)dx = I_6 + I_7. \tag{24}$$

From (23), we have

$$I_4 = \frac{-c(c-1)}{2b}\int_{-\infty}^{0}\left(\frac{-x+a}{b}\right)^{c-1}\exp\left[-\left(\frac{-x+a}{b}\right)^c\right]\ln\left(\frac{-x+a}{b}\right)dx, \tag{25}$$

$$I_5 = \frac{-c(c-1)}{2b}\int_{0}^{\infty}\left(\frac{x-a}{b}\right)^{c-1}\exp\left[-\left(\frac{x-a}{b}\right)^c\right]\ln\left(\frac{x-a}{b}\right)dx. \tag{26}$$

Substituting $(-x+a)^c b^{-c}=u$, formula (25) takes on the form of

$$I_4=\frac{(c-1)}{2c}\int_{-\infty}^{0}\exp(-u)\ln(u)du=\frac{(c-1)}{2c}\gamma. \tag{27}$$

Substituting $(x-a)^c b^{-c}=u$, formula (26) takes on the form of

$$I_5=\frac{-(c-1)}{2c}\int_{0}^{\infty}\exp(-u)\ln(u)du=\frac{(c-1)}{2c}\gamma. \tag{28}$$

Formula (23), based on (27) and (28), is in the form of

$$I_2=I_4+I_5=\frac{(c-1)}{2c}\gamma+\frac{(c-1)}{2c}\gamma=\frac{(c-1)}{2}\gamma. \tag{29}$$

From (24), we have

$$I_6=\frac{c}{2b}\int_{-\infty}^{0}\left(\frac{-x+a}{b}\right)^c\exp\left[-\left(\frac{-x+a}{b}\right)^c\right]dx, \tag{30}$$

$$I_7=\frac{c}{2b}\int_{0}^{\infty}\left(\frac{x-a}{b}\right)^c\exp\left[-\left(\frac{x-a}{b}\right)^c\right]dx. \tag{31}$$

Substituting $(-x+a)^c b^{-c}=u$, formula (30) takes on the form of

$$I_6=-0.5\int_{-\infty}^{0}u^{\frac{1}{c}}\exp(-u)du=0.5\Gamma\left(\frac{1}{c}+1\right). \tag{32}$$

Substituting $(x-a)^c b^{-c}=u$, formula (31) is given by

$$I_7=0.5\int_{0}^{\infty}u^{1/c}\exp(-u)du=0.5\Gamma\left(\frac{1}{c}+1\right). \tag{33}$$

Formula (24), based on (32) and (33), is

$$I_3=I_6+I_7=0.5\Gamma\left(\frac{1}{c}+1\right)+0.5\Gamma\left(\frac{1}{c}+1\right)=\Gamma\left(\frac{1}{c}+1\right). \tag{34}$$

Substituting (22), (29) and (34) for (21), we get (17).
The Tsallis (T) entropy of order α is defined as (Tabass et al., 2016; Tsallis, 1988)

$$T_\alpha(a,b,c)=\frac{1}{\alpha-1}\int_{-\infty}^{\infty}f(x;\mathrm{a},\mathrm{b},\mathrm{c})^\alpha dx-1\ (\alpha>0,\alpha\neq 1),$$

Then, based on (2), it transforms into

$$T_\alpha(a,b,c) = \frac{c^\alpha}{(\alpha-1)\,(2b)^\alpha} Int(\alpha,a,b,c) - 1.$$

where $Int(\alpha,a,b,c)$ is given by (20).

Renyi (R) entropy of order α is defined as (Tabass et al., 2016)

$$R_\alpha(a,b,c) = \tfrac{1}{1-\alpha} \ln \int_{-\infty}^{\infty} f(x;a,b,c)^\alpha dx \; (\alpha > 0, \alpha \neq 1).$$

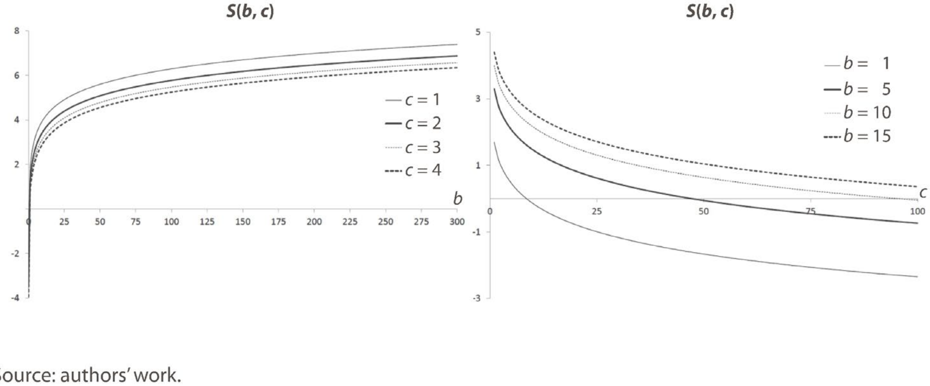
Then, based on (2), it transforms into

$$R_\alpha(a,b,c) = \frac{1}{(1-\alpha)} \ln\left[\left(\frac{c}{2b}\right)^\alpha \int_{-\infty}^{\infty} \left|\frac{x_1-a}{b}\right|^{\alpha(c-1)} \exp\left(-\left|\frac{x_1-a}{b}\right|^c\right)^\alpha dx_1\right] =$$
$$= \frac{\alpha}{(1-\alpha)} \ln\left(\frac{c}{2b}\right) + \frac{1}{(1-\alpha)} \text{ In } [Int(\alpha,a,b,c)],$$

where $Int(\alpha,a,b,c)$ is given by (20). The proof is complete.

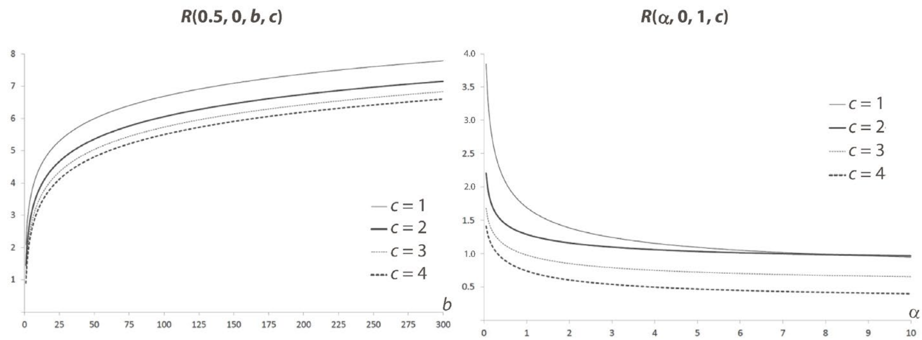
The Renyi and Tsallis entropies converge to the Shannon entropy. Figures 4–6 show the Shannon, Renyi and Tsallis entropies of the $BPL(a,b,c)$ distribution. The Shannon entropy as a function of b increases, especially for the initial values of b. The same entropy as a function of c decreases, especially for the initial values of c. The Renyi entropy as a function of b increases strictly, especially for the initial values of b. The smaller the values of c, the higher the values of the Renyi entropy. The Renyi entropy as a function of α decreases strictly, especially for the initial values of α. The Tsallis entropy as a function of b decreases strictly. The higher the values of c, the higher the values of the Tsallis entropy. The Tsallis entropy as a function of α shows a maximum for $\alpha < 1$ and decreases for $\alpha > 1$, especially for the initial values of α. If $\alpha \to \infty$, then $T \to -1$ and parameter c does not affect the values of T.

Figure 4. Shannon entropy of the *BPL*(*a*, *b*, *c*) distribution



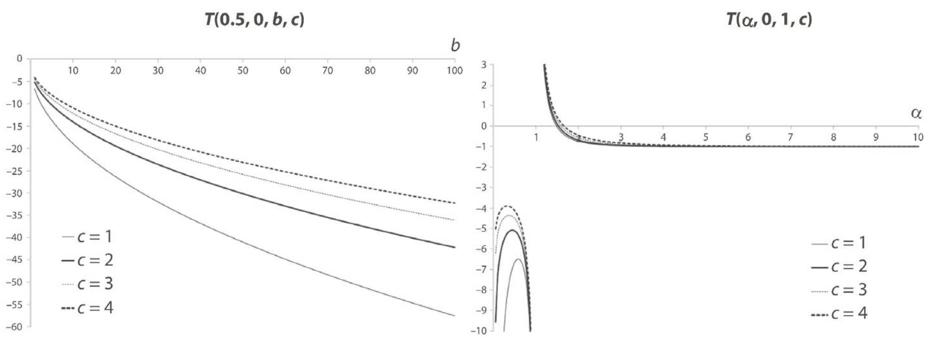
Source: authors' work.

Figure 5. Renyi entropy of the *BPL*(*a*, *b*, *c*) distribution



Source: authors' work.

Figure 6. Tsallis entropy of the *BPL*(*a*, *b*, *c*) distribution



Source: authors' work.

2.7. Fisher information matrix

The Fisher information matrix plays an important role in many statistical applications. The Fisher information matrix is used e.g. to construct hypothesis tests and confidence intervals using maximum likelihood estimators in the frequentist paradigm, to define the default priority in the Bayesian paradigm, or to measure model complexity in the minimum description length paradigm (Ly et al., 2017).

Theorem 10. Fisher information matrix $I_{i,j}$ $(i, j = 1, 2, 3)$ for the $BPL(a, b, c)$ distribution is given by

$$I_{11} = \frac{\frac{\Gamma\left(1-\frac{2}{c}\right)}{b^2}}{(c-1)^{-2}} - \frac{b^c}{\left(\frac{2c^2-2c}{b^c}\right)^{-1}} + \frac{2b^{2c}}{c^{-2}b^{2c}} = \frac{(c-1)^2\Gamma\left(1-\frac{2}{c}\right)}{b^2} + 2c,$$

$$I_{12}=I_{21}=\frac{(2c\mathrm{sgn}(b,0)-\mathrm{sgn}(b,0)-c^2\mathrm{sgn}(b,0)-c+1)}{b^2\Gamma\left(1-\frac{1}{c}\right)^{-1}}+$$
$$+\frac{c\Gamma\left(2-\frac{1}{c}\right)}{b+b^{1-c}}-\frac{c^2\mathrm{sgn}(b,0)}{b^2\Gamma\left(3-\frac{1}{c}\right)^{-1}}+\frac{\mathrm{sgn}(b,0)(2c^2-2c)}{b^2\Gamma\left(2-\frac{1}{c}\right)^{-1}},$$

$$I_{13}=I_{31}=\frac{\Gamma\left(1-\frac{1}{c}\right)\left(\frac{1}{c}-1\right)}{b[1-c\ln(b)]^{-1}}+\frac{\Gamma\left(2-\frac{1}{c}\right)}{b[1+\ln(b)-2c\ln(b)]^{-1}}+\frac{c\ln(b)}{b\Gamma\left(3-\frac{1}{c}\right)^{-1}}+$$
$$+\frac{\Gamma\left(1-\frac{1}{c}\right)(1-c)}{b^{2c-1}\left[c\ln(b)+\Psi\left(1-\frac{1}{c}\right)\right]^{-1}}+$$
$$+\frac{\Gamma\left(2-\frac{1}{c}\right)(2c+1)}{b^{3c-1}\left[c\ln(b)+\Psi\left(2-\frac{1}{c}\right)\right]^{-1}}-\frac{\mathrm{c}\left[c\ln(b)+\Psi\left(3-\frac{1}{c}\right)\right]}{b^{4c-1}\Gamma\left(3-\frac{1}{c}\right)^{-1}},$$

$$I_{22}=\frac{[c\mathrm{sgn}(b,0)-\mathrm{sgn}(b,0)+1]^2}{b^3}+\frac{2c\mathrm{sgn}(b,0)[\mathrm{sgn}(b,0)-1]}{b^{-3}},$$

$$I_{23}=I_{32}=\frac{\Gamma\left(\frac{1}{c}\right)\left[c+\Psi\left(\frac{1}{c}\right)\right][1-\mathrm{sgn}(b,0)+c\mathrm{sgn}(b,0)]}{bc^2}+$$
$$+\frac{[2\mathrm{sgn}(b,0)-c\mathrm{sgn}(b,0)-2]\Gamma\left(\frac{1}{c}\right)\Psi\left(\frac{1}{c}\right)}{bc},$$

$$I_{33}=\frac{2\Gamma\left(\frac{1}{c}\right)\Psi\left(\frac{1}{c}\right)+\frac{\pi^2}{6}+4\gamma+3}{c^2}-\frac{2\Gamma\left(\frac{1}{c}\right)\left[\mathrm{c}+\Psi\left(\frac{1}{c}\right)\right]}{3},$$

where $\gamma\approx 0.577$ is the Euler-Mascheroni constant, $\mathrm{sgn}(z,x)=0\cdot I(z=0)+$ $+z/|z|\cdot I(z\neq 0)$ and $\Psi(x)=\frac{d}{dx}\ln\Gamma(x),\Psi'(x)=\frac{d^2}{dx^2}\ln\Gamma(x)$ are digamma and digamma1 functions, respectively.

Proof. First, we need to calculate the logarithm. From (2), we have

$$\ln[f(x_1|a,b,c)]=\ln\left(\frac{c}{2b}\right)+(c-1)\ln\left(\left|\frac{x_1-a}{b}\right|\right)-\left|\frac{x_1-a}{b}\right|^c.$$

Then, we need to calculate the partial derivatives

$$\frac{d\ln[f(x_1|a,b,c)]}{da} = \frac{\mathrm{sgn}(x_1-a,0)\left[c\left|\frac{x_1-a}{b}\right|^c - c + 1\right]}{|x_1-a|},$$

$$\frac{d\ln[f(x_1|a,b,c)]}{db} = b^{-1}c\left(\left|\frac{x_1-a}{b}\right|^c - 1\right),$$

$$\frac{d\ln[f(x_1|a,b,c)]}{dc} = \ln\left|\frac{x_1-a}{b}\right| - \left|\frac{x_1-a}{b}\right|^c \ln\left|\frac{x_1-a}{b}\right| + \frac{1}{c}.$$

Thus, we obtain the Fisher score in the form of

$$\boldsymbol{h}(x_1;a,b,c) = \begin{bmatrix} \frac{\mathrm{sgn}(x_1-a,0)\left[c\left|\frac{x_1-a}{b}\right|^c - c+1\right]}{|x_1-a|} \\ b^{-1}c\left(\left|\frac{x_1-a}{b}\right|^c - 1\right) \\ \ln\left|\frac{x_1-a}{b}\right| - \left|\frac{x_1-a}{b}\right|^c \ln\left|\frac{x_1-a}{b}\right| + \frac{1}{c} \end{bmatrix}.$$

Let $\boldsymbol{u}(x_1;a,p) = \boldsymbol{h}(x_1;a,p)\boldsymbol{h}(x_1;a,p)^{\mathrm{T}}$, then

$$u_{11} = \frac{\left[c\left|\frac{x_1-a}{b}\right|^c - c + 1\right]^2}{(x_1-a)^2}, \tag{35}$$

$$u_{12} = u_{21} = \frac{\left[c\left|\frac{x_1-a}{b}\right|^c - c + 1\right]\begin{bmatrix} 1 - \mathrm{sgn}(b,0) + c\mathrm{sgn}(b,0) \\ -c\left|\frac{x_1-a}{b}\right|^c c\mathrm{sgn}(b,0) \end{bmatrix}}{|x_1-a|b}, \tag{36}$$

$$u_{13} = u_{31} = \frac{\left[\frac{c}{b}\left|\frac{x_1-a}{b}\right|^c - \frac{c}{b} + \frac{1}{b}\right]\left[\ln\left|\frac{x_1-a}{b}\right| - \left|\frac{x_1-a}{b}\right|^c \ln\left|\frac{x_1-a}{b}\right| + \frac{1}{c}\right]}{\left|\frac{x_1-a}{b}\right|}, \tag{37}$$

$$u_{22} = \frac{\left[1 - \mathrm{sgn}(b,0) + c\mathrm{sgn}(b,0) - c\mathrm{sgn}(b,0)\left|\frac{x_1-a}{b}\right|^c\right]^2}{b^3}, \tag{38}$$

$$u_{23} = u_{32} = \frac{\left[-\ln\left|\frac{x_1-a}{b}\right| + \left|\frac{x_1-a}{b}\right|^c \ln\left|\frac{x_1-a}{b}\right| - \frac{1}{c}\right]\left[1 - \mathrm{sgn}(b,0) + c\mathrm{sgn}(b,0) - c\mathrm{sgn}(b,0)\left|\frac{x_1-a}{b}\right|^c\right]}{b}, \tag{39}$$

$$u_{33} = \left[\ln\left|\frac{x_1-a}{b}\right| - \left|\frac{x_1-a}{b}\right|^c \ln\left|\frac{x_1-a}{b}\right| + \frac{1}{c}\right]^2. \tag{40}$$

Let $I_{i,j}$ $(i, j = 1, 2, 3) = E[u_{ij}]$, then based on (35)–(40), we have

$$I_{11} = \frac{E\left[\frac{1}{|x_1 - a|^2}\right]}{(c-1)^{-2}} - \frac{E\left[\frac{1}{|x_1 - a|^{-c}}\right]}{\left(\frac{2c^2 - 2c}{b^c}\right)^{-1}} + \frac{E\left[\frac{1}{|x_1 - a|^{-2c}}\right]}{c^{-2} b^{2c}}, \tag{41}$$

$$\begin{gathered} I_{12} = I_{21} = \frac{2c\mathrm{sgn}(b,0) - \mathrm{sgn}(b,0) - c^2\mathrm{sgn}(b,0) - c + 1}{+bE\left[\frac{1}{|x_1 - a|}\right]^{-1}} + \\ + \frac{cE\left[\frac{1}{|x_1 - a|^{1-c}}\right]}{(b^c + 1)^{-1}} + \frac{E\left[\frac{1}{|x_1 - a|^{1-2c}}\right]}{b^{2c+1} c^{-2} \mathrm{sgn}(b,0)^{-1}} + \frac{\mathrm{sgn}(b,0) E\left[\frac{1}{|x_1 - a|^{1-c}}\right]}{b^{c+1}(2c^2 - 2c)^{-1}}, \end{gathered} \tag{42}$$

$$\begin{gathered} I_{13} = I_{31} = \frac{E\left[\frac{1}{|x_1 - a|}\right]}{\left\{\left(\frac{1}{c} - 1\right)[1 - c\ln(b)]\right\}^{-1}} + \frac{E\left[\frac{\ln|x_1 - a|}{|x_1 - a|}\right]}{(1-c)^{-1}} + \\ + \frac{b^{-c} E\left[\frac{1}{|x_1 - a|^{1-c}}\right]}{[1 + \ln(b) - 2c\ln(b)]^{-1}} + \frac{E\left[\frac{\ln|x_1 - a|}{|x_1 - a|^{1-c}}\right]}{b^c (2c+1)^{-1}} - \frac{E\left[\frac{\ln|x_1 - a|}{|x_1 - a|^{1-2c}}\right]}{c^{-1} b^{2c}} + \\ + \frac{E\left[\frac{1}{|x_1 - a|^{1-2c}}\right]}{b^{2c}[c\ln(b)]^{-1}}, \end{gathered} \tag{43}$$

$$\begin{gathered} I_{22} = \frac{[c\mathrm{sgn}(b,0) - \mathrm{sgn}(b,0) + 1]^2}{b^3} + \frac{2cE\left[\frac{1}{|x_1 - a|^{-c}}\right] b - c\,\mathrm{sgn}(b,0)}{b^3[\mathrm{sgn}(b,0) - c\mathrm{sgn}(b,0) - 1]^{-1}} + \\ + \frac{E\left[\frac{1}{|x_1 - a|^{-2c}}\right]}{c^{-2} b^{2c+3} \mathrm{sgn}(b,0)^{-2}}, \end{gathered} \tag{44}$$

$$\begin{gathered} I_{23} = I_{32} = \frac{\mathrm{sgn}(b,0) - c\mathrm{sgn}(b,0) - 1}{bc} + \frac{E\left[\frac{1}{|x_1 - a|^{-c}}\right]}{b^{c+1}\mathrm{sgn}(b,0)^{-1}} + \\ + \frac{E\left[\left|\frac{x_1 - a}{b}\right|^c \ln\left|\frac{x_1 - a}{b}\right|\right] - E\left[\ln\left|\frac{x_1 - a}{b}\right|\right]}{b[1 - \mathrm{sgn}(b,0) + c\mathrm{sgn}(b,0)]^{-1}}, \end{gathered} \tag{45}$$

$$\begin{gathered} I_{33} = E\left[\left(\ln\left|\frac{x_1 - a}{b}\right|\right)^2\right] + \frac{2E\left[\ln\left|\frac{x_1 - a}{b}\right|\right] + \frac{1}{c}}{c} + \\ + E\left[\frac{\left|\frac{x_1 - a}{b}\right|^{2c}}{\left(\ln\left|\frac{x_1 - a}{b}\right|\right)^{-2}}\right] - \frac{2}{c} E\left[\frac{\left|\frac{x_1 - a}{b}\right|^c}{\left(\ln\left|\frac{x_1 - a}{b}\right|\right)^{-1}}\right] - 2E\left[\frac{\left|\frac{x_1 - a}{b}\right|^c}{\left(\ln\left|\frac{x_1 - a}{b}\right|\right)^{-2}}\right]. \end{gathered} \tag{46}$$

Let us define the following:

$$Int_A(d,e) = \frac{c}{2b^c}\int_{-\infty}^{\infty} |x_1 - a|^{cd+e} \exp\left(-\left|\frac{x_1 - a}{b}\right|^c\right) dx_1, \tag{47}$$

$$Int_B(d,e) = \frac{c}{2b}\int_{-\infty}^{\infty} \left|\frac{x_1 - a}{b}\right|^{cd+e} \ln\left|\frac{x_1 - a}{b}\right| \exp\left(-\left|\frac{x_1 - a}{b}\right|^c\right) dx_1, \tag{48}$$

$$Int_C(d,e) = \frac{c}{2b^c}\int_{-\infty}^{\infty} |x_1 - a|^{cd+e} \ln|x_1 - a| \exp\left(-\left|\frac{x_1 - a}{b}\right|^c\right) dx_1, \tag{49}$$

$$Int_D(d,e) = \frac{c}{2b}\int_{-\infty}^{\infty} \left|\frac{x_1 - a}{b}\right|^{cd+e} \left(\ln\left|\frac{x_1 - a}{b}\right|\right)^2 \exp\left(-\left|\frac{x_1 - a}{b}\right|^c\right) dx_1. \tag{50}$$

Substituting $|x_1 - a|^c b^{-c} = u$, we write (47)–(50) in a simpler form, namely:

$$\begin{aligned} Int_A(d,e) &= b^{cd+e-1+c}\int_0^{\infty} u^{\frac{1}{c}+d+\frac{e}{c}-1} \exp(-u) du = \\ &= b^{cd+e-1+c}\Gamma\left(\frac{1}{c} + d + \frac{e}{c}\right), \end{aligned} \tag{51}$$

$$\begin{aligned} Int_B(d,e) &= \frac{1}{c}\int_0^{\infty} u^{\frac{1}{c}+d+\frac{e}{c}-1} \ln(u) \exp(-u) du = \\ &= \frac{\Gamma\left(\frac{1}{c} + d + \frac{e}{c}\right)\Psi\left(\frac{1}{c} + d + \frac{e}{c}\right)}{c}, \end{aligned} \tag{52}$$

$$\begin{aligned} Int_C(d,e) &= \frac{c}{b^{2c-1}}\int_0^{\infty} u^{\frac{1}{c}+d+\frac{e}{c}-1} \ln\left(bu^{\frac{1}{c}}\right) \exp(-u) du = \\ &= \frac{\Gamma\left(\frac{1}{c} + d + \frac{e}{c}\right)\left[\Psi\left(\frac{1}{c} + d + \frac{e}{c}\right) + c\ln(b)\right]}{b^{2c-1}} \end{aligned} \tag{53}$$

$$\begin{aligned} Int_D(d,e) &= \frac{1}{c^2}\int_0^{\infty} u^{\frac{1}{c}+d+\frac{e}{c}-1} [\ln(u)]^2 \exp(-u) du = \\ &= \frac{\left\{\left[\Psi\left(\frac{1}{c} + d + \frac{e}{c}\right)\right]^2 + \Psi'\left(\frac{1}{c} + d + \frac{e}{c}\right)\right\}}{\Gamma\left(\frac{1}{c} + d + \frac{e}{c}\right)^{-1} c^2} \end{aligned} \tag{54}$$

where $\frac{1}{c} + d + \frac{e}{c} > 0$.

To write the Fisher Information Matrix in a simpler form, we need to calculate formulas Int_i $(i = 1, \dots, 12)$ as shown below. Based on (51)–(54), we use

$Int_A(d,e), Int_B(d,e)$, $Int_C(d,e)$ and $Int_D(d,e)$ to calculate Int_{1-6}, Int_{7-8}, Int_{9-11} and Int_{12-14}, respectively. We obtain

$$Int_1 = E\left[\frac{1}{|x_1-a|^2}\right] = \frac{c}{2b^c}\int_{-\infty}^{\infty} \frac{|x_1-a|^{c-3}}{\exp\left(-\left|\frac{x_1-a}{b}\right|^c\right)^{-1}} dx_1 = \\ = Int_A(1,-3) = \frac{\Gamma\left(1-\frac{2}{c}\right)}{b^2},$$

$$Int_2 = E\left[\frac{1}{|x_1-a|^{-c}}\right] = \frac{c}{2b^c}\int_{-\infty}^{\infty} \frac{|x_1-a|^{2c-1}}{\exp\left(-\left|\frac{x_1-a}{b}\right|^c\right)^{-1}} dx_1 = Int_A(2,-1) = b^c,$$

$$Int_3 = E\left[\frac{1}{|x_1-a|^{-2c}}\right] = \frac{c}{2b^c}\int_{-\infty}^{\infty} \frac{|x_1-a|^{3c-1}}{\exp\left(-\left|\frac{x_1-a}{b}\right|^c\right)^{-1}} dx_1 = Int_A(3,-1) = 2b^{2c},$$

$$Int_4 = E\left[\frac{1}{|x_1-a|}\right] = \\ = \frac{c}{2b^c}\int_{-\infty}^{\infty} \frac{|x_1-a|^{c-2}}{\exp\left(-\left|\frac{x_1-a}{b}\right|^c\right)^{-1}} dx_1 = Int_A(1,-2) = \frac{\Gamma\left(1-\frac{1}{c}\right)}{b},$$

$$Int_5 = E\left[\frac{1}{|x_1-a|^{1-c}}\right] = \frac{c}{2b^c}\int_{-\infty}^{\infty} \frac{|x_1-a|^{2c-2}}{\exp\left(-\left|\frac{x_1-a}{b}\right|^c\right)^{-1}} dx_1 = Int_A(2,-2) = \\ = \frac{\Gamma\left(2-\frac{1}{c}\right)}{b^{1-c}},$$

$$Int_6 = E\left[\frac{1}{|x_1-a|^{1-2c}}\right] = \\ = \frac{c}{2b^c}\int_{-\infty}^{\infty} \frac{|x_1-a|^{3c-2}}{\exp\left(-\left|\frac{x_1-a}{b}\right|^c\right)^{-1}} dx_1 = Int_A(3,-2) = \frac{\Gamma\left(3-\frac{1}{c}\right)}{b^{1-2c}},$$

$$Int_7 = E\left[\ln\left|\frac{x_1-a}{b}\right|\right] = \frac{c}{2b}\int_{-\infty}^{\infty} \frac{\left|\frac{x_1-a}{b}\right|^{c-1}}{\exp\left(-\left|\frac{x_1-a}{b}\right|^c\right)^{-1}} \ln\left|\frac{x_1-a}{b}\right| dx_1 = \\ = Int_B(0,0) = \frac{\Gamma\left(\frac{1}{c}\right)\Psi\left(\frac{1}{c}\right)}{c},$$

$$Int_8 = E\left[\frac{\ln\left|\frac{x_1-a}{b}\right|}{\left|\frac{x_1-a}{b}\right|^{-c}}\right] = \frac{c}{2b}\int_{-\infty}^{\infty}\frac{\left|\frac{x_1-a}{b}\right|^{2c-1}}{\exp\left(-\left|\frac{x_1-a}{b}\right|^{c}\right)^{-1}}\ln\left|\frac{x_1-a}{b}\right|dx_1 =$$
$$= Int_B(1,0) = \frac{\Gamma\left(\frac{1}{c}\right)\left[c+\Psi\left(\frac{1}{c}\right)\right]}{c^2},$$

$$Int_9 = E\left[\frac{\ln|x_1-a|}{|x_1-a|}\right] = \frac{c}{2b^{2c}}\int_{-\infty}^{\infty}\frac{|x_1-a|^{c-2}\ln|x_1-a|}{\exp\left(-\left|\frac{x_1-a}{b}\right|^{c}\right)^{-1}}dx_1 = Int_C(1,-2) =$$
$$= \frac{\Gamma\left(1-\frac{1}{c}\right)\left[c\ln(b)+\Psi\left(1-\frac{1}{c}\right)\right]}{b^{2c-1}},$$

$$Int_{10} = E\left[\frac{\ln|x_1-a|}{|x_1-a|^{1-c}}\right] = \frac{0.5c}{b^{2c}}\int_{-\infty}^{\infty}\frac{|x_1-a|^{2c-2}\ln|x_1-a|}{\exp\left(-\left|\frac{x_1-a}{b}\right|^{c}\right)^{-1}}dx_1 = Int_C(2,-2) =$$
$$= \frac{\Gamma\left(2-\frac{1}{c}\right)\left[c\ln(b)+\Psi\left(2-\frac{1}{c}\right)\right]}{b^{2c-1}},$$

$$Int_{11} = E\left[\frac{\ln|x_1-a|}{|x_1-a|^{1-2c}}\right] = \frac{c}{2b^{2c}}\int_{-\infty}^{\infty}\frac{|x_1-a|^{3c-2}\ln|x_1-a|}{\exp\left(-\left|\frac{x_1-a}{b}\right|^{c}\right)^{-1}}dx_1 = Int_C(3,-2) =$$
$$= \frac{\Gamma\left(3-\frac{1}{c}\right)\left[c\ln(b)+\Psi\left(3-\frac{1}{c}\right)\right]}{b^{2c-1}},$$

$$Int_{12} = E\left[\left(\ln\left|\frac{x_1-a}{b}\right|\right)^2\right] =$$
$$= \frac{c}{2b}\int_{-\infty}^{\infty}\frac{\left|\frac{x_1-a}{b}\right|^{c-1}\left(\ln\left|\frac{x_1-a}{b}\right|\right)^2}{\exp\left(-\left|\frac{x_1-a}{b}\right|^{c}\right)^{-1}}dx_1 = Int_D(1,-1) = \frac{6\gamma+\pi^2}{6c^2},$$

$$Int_{13} = E\left[\frac{\left(\ln\left|\frac{x_1-a}{b}\right|\right)^2}{\left|\frac{x_1-a}{b}\right|^{-2c}}\right] =$$
$$= \frac{c}{2b}\int_{-\infty}^{\infty}\frac{\left|\frac{x_1-a}{b}\right|^{3c-1}\left(\ln\left|\frac{x_1-a}{b}\right|\right)^2}{\exp\left(-\left|\frac{x_1-a}{b}\right|^{c}\right)^{-1}}dx_1 = Int_D(3,-1) =$$
$$= \frac{2\gamma^2-6\gamma+\frac{\pi^2}{3}+2}{c^2},$$

$$Int_{14} = E\left[\frac{\left(\ln\left|\frac{x_1-a}{b}\right|\right)^2}{\left|\frac{x_1-a}{b}\right|^{-c}}\right] =$$
$$= \frac{c}{2b}\int_{-\infty}^{\infty}\frac{\left|\frac{x_1-a}{b}\right|^{2c-1}\left(\ln\left|\frac{x_1-a}{b}\right|\right)^2}{\exp\left(-\left|\frac{x_1-a}{b}\right|^c\right)^{-1}}dx_1 = Int_D(2,-1) =$$
$$= \frac{\gamma^2 - 2\gamma + \frac{\pi^2}{6}}{c^2}.$$

Substituting formulas Int_1–Int_{14} for (41)–(46), as a result of simple transformations, we receive

$$I_{11} = \frac{\frac{\Gamma\left(1-\frac{2}{c}\right)}{b^2}}{(c-1)^{-2}} - \frac{b^c}{\left(\frac{2c^2-2c}{b^c}\right)^{-1}} + \frac{2b^{2c}}{c^{-2}b^{2c}} = \frac{(c-1)^2\Gamma\left(1-\frac{2}{c}\right)}{b^2} + 2c,$$

$$I_{12} = \frac{(2c\mathrm{sgn}(b,0) - \mathrm{sgn}(b,0) - c^2\mathrm{sgn}(b,0) - c + 1)}{b^2\Gamma\left(1-\frac{1}{c}\right)^{-1}} +$$
$$+\frac{c\Gamma\left(2-\frac{1}{c}\right)}{b+b^{1-c}} - \frac{c^2\mathrm{sgn}(b,0)}{b^2\Gamma\left(3-\frac{1}{c}\right)^{-1}} + \frac{\mathrm{sgn}(b,0)(2c^2-2c)}{b^2\Gamma\left(2-\frac{1}{c}\right)^{-1}},$$

$$I_{13} = \frac{\Gamma\left(1-\frac{1}{c}\right)\left(\frac{1}{c}-1\right)}{b[1-c\ln(b)]^{-1}} + \frac{\Gamma\left(2-\frac{1}{c}\right)}{b[1+\ln(b)-2c\ln(b)]^{-1}} + \frac{c\ln(b)}{b\Gamma\left(3-\frac{1}{c}\right)^{-1}} +$$
$$+\frac{\Gamma\left(1-\frac{1}{c}\right)(1-c)}{b^{2c-1}\left[c\ln(b)+\Psi\left(1-\frac{1}{c}\right)\right]^{-1}} +$$
$$+\frac{\Gamma\left(2-\frac{1}{c}\right)(2c+1)}{b^{3c-1}\left[c\ln(b)+\Psi\left(2-\frac{1}{c}\right)\right]^{-1}} - \frac{\mathrm{c}\left[c\ln(b)+\Psi\left(3-\frac{1}{c}\right)\right]}{b^{4c-1}\Gamma\left(3-\frac{1}{c}\right)^{-1}},$$

$$I_{22} = \frac{[c\mathrm{sgn}(b,0) - \mathrm{sgn}(b,0) + 1]^2}{b^3} + \frac{2c\mathrm{sgn}(b,0)[\mathrm{sgn}(b,0)-1]}{b^{-3}},$$

$$I_{23} = I_{32} = \frac{\Gamma\left(\frac{1}{c}\right)\left[c+\Psi\left(\frac{1}{c}\right)\right][1-\mathrm{sgn}(b,0)+c\mathrm{sgn}(b,0)]}{bc^2} +$$
$$+\frac{[2\mathrm{sgn}(b,0) - c\mathrm{sgn}(b,0) - 2]\Gamma\left(\frac{1}{c}\right)\Psi\left(\frac{1}{c}\right)}{bc},$$

$$I_{33} = \frac{2\Gamma\left(\frac{1}{c}\right)\Psi\left(\frac{1}{c}\right) + \frac{\pi^2}{6} + 4\gamma + 3}{c^2} - \frac{2\Gamma\left(\frac{1}{c}\right)\left[c + \Psi\left(\frac{1}{c}\right)\right]}{3}.$$

The proof is thus complete.

3. Estimation methods and simulation study

Studying the literature devoted to the distribution theory, we find that the most dominant method of parameter estimation is the maximum likelihood (ML) method. However, the question remains whether this choice is right. The more recent the paper, the more often the parameters are estimated using other methods, e.g. the ordinary least-squares (OLS) and weighted least-squares (WLS) ones (see e.g. Almetwally, 2022; Shama et al., 2023; Sulewski et al., 2025). The natural estimation measure is also a very interesting proposal, involving the use of the least absolute values (LAV). The LAV measure corresponds to the absolute values of the differences between the empirical and theoretical CDFs (Sulewski et al., 2025).

Let $\boldsymbol{\vartheta} = (a, b, c)$ be the parameter vector and $x_1^*,\ x_2^*,\ \dots,\ x_n^*$ be a random sample of size n from the $BPL(a, b, c)$. To estimate the unknown values of the parameters, we use such estimation methods as the ML, OLS, WLS and LAV. All calculations were performed using the R software.

The likelihood function (LF), based on (2), is given by

$$L(x_i^*; \boldsymbol{\vartheta}) = \frac{c}{2b}\prod_{i=1}^{n}\left|\frac{x_i^*-a}{b}\right|^{c-1}\exp\left(-\left|\frac{x_i^*-a}{b}\right|^{c}\right), \tag{55}$$

thus, the log-likelihood function (LLF) is defined as

$$l(x_i^*; \boldsymbol{\vartheta}) = \ln L(x_i^*; \boldsymbol{\vartheta}) = n\ln\frac{c}{2b} + (c-1)\sum_{i=1}^{n}\ln\left|\frac{x_i^*-a}{b}\right| - \sum_{i=1}^{n}\left|\frac{x_i^*-a}{b}\right|^{c}. \tag{56}$$

Formulas $\frac{dl}{da}, \frac{dl}{db}, \frac{dl}{dc}$ have complex forms, so in practice, to obtain the ML estimates, we maximise the LF (55) or the LLF (56) e.g. in Excel, the R software and Mathcad instead of struggling with a system of three complicated nonlinear equations that may have extraneous roots.

To obtain the OLS, WLS and LAV estimates of the BPL parameters, we minimise the following objective functions, respectively:

$$OLS(x_i^*; \boldsymbol{\vartheta}) = \sum_{i=1}^{n}\left\{\frac{i}{n+1} - F(x_i^*; \boldsymbol{\vartheta})\right\}^2, \tag{57}$$

$$WLS(x_i^*;\boldsymbol{\vartheta}) = \sum\nolimits_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left\{\frac{i}{n+1} - F(x_i^*;\boldsymbol{\vartheta})\right\}^2, \tag{58}$$

$$LAV(x_i^*;\boldsymbol{\vartheta}) = \sum\nolimits_{i=1}^{n} \left|\frac{i}{n+1} - F(x_i^*;\boldsymbol{\vartheta})\right|^2, \tag{59}$$

where $F(x_i^*;\boldsymbol{\vartheta})$ is given by (7).

The biases and the root mean squared errors (RMSEs) of the MLEs are shown in Table 1. The simulation study was performed with 10^4 samples using sample sizes of $50, 100, 200$. The samples were drawn from the $BPL(0,1,\mathrm{c})$, where $c = (1,2,3)$. To obtain highly accurate parameter estimates, the optimisation procedure was run 10^2 times with $a_s = Unif(a-0.1, a+0.1)$, $b_s = Unif(b, b+0.1)$ and $c_s = Unif(c, c+0.1)$ random initial values. Our estimates for a given sample are the values that minimise (57)–(59) and maximise (56).

We observe that the estimates approach true values when the sample size increases, which implies the consistency of the estimates. The RMSE decreases along with the value of c for $\hat{a}, \hat{b}$ and increases with the value of c for $\hat{c}$. The bias increases with the value of c for $\hat{c}$. The smallest bias is for the position parameter.

Table 1. Biases and RMSEs of the estimates obtained through various methods (M)

M	$\hat{a}$		$\hat{b}$		$\hat{c}$	
	bias	RMSE	bias	RMSE	bias	RMSE
			$c = 1, n = 50$			
ML	0.0017	0.1682	−0.0171	0.1540	0.0080	0.1632
OLS	0.0018	0.1617	0.0385	0.1674	0.0059	0.1566
WLS	−0.0022	0.1641	0.0196	0.1589	0.0159	0.1523
LAV	−0.0020	0.1607	0.0400	0.1679	0.0079	0.1664
			$c = 1, n = 100$			
ML	−0.0013	0.1072	−0.0073	0.1077	−0.0013	0.1037
OLS	0.0010	0.1108	0.0172	0.1145	0.0024	0.1031
WLS	−0.0014	0.1138	0.0086	0.1096	0.0094	0.0969
LAV	0.0002	0.1119	0.0195	0.1169	0.0034	0.1076
			$c = 1, n = 200$			
ML	−0.0004	0.0811	−0.0054	0.0770	−0.0011	0.0698
OLS	0.0007	0.0774	0.0092	0.0799	0.0010	0.0697
WLS	−0.0003	0.0794	0.0038	0.0770	0.0055	0.0667
LAV	−0.0002	0.0790	0.0105	0.0811	0.0024	0.0731
			$c = 2, n = 50$			
ML	0.0004	0.0641	0.0008	0.0750	0.0807	0.2534
OLS	−0.0003	0.1334	0.0074	0.0823	0.0216	0.3287
WLS	−0.0006	0.1230	0.0018	0.0797	0.0298	0.3038
LAV	0.0027	0.1248	0.0086	0.0831	0.0414	0.3562
			$c = 2, n = 100$			
ML	−0.0003	0.0390	0.0005	0.0523	0.0422	0.1667
OLS	0.0002	0.0912	0.0048	0.0561	0.0137	0.2063
WLS	−0.0004	0.0850	0.0017	0.0546	0.0225	0.1948
LAV	−0.0004	0.0863	0.0045	0.0573	0.0184	0.2242

Table 1. Biases and RMSEs of the estimates obtained through various methods (M; cont.)

M	$\hat{a}$		$\hat{b}$		$\hat{c}$	
	bias	RMSE	bias	RMSE	bias	RMSE
			$c = 2, n = 200$			
ML	0.0000	0.0284	−0.0004	0.0373	0.0188	0.1143
OLS	−0.0001	0.0645	0.0015	0.0396	0.0061	0.1401
WLS	−0.0001	0.0585	0.0004	0.0386	0.0136	0.1309
LAV	−0.0003	0.0612	0.0026	0.0405	0.0124	0.1501
			$c = 3, n = 50$			
ML	0.0004	0.0448	−0.0019	0.0498	0.1214	0.3797
OLS	0.0013	0.0882	0.0040	0.0547	0.0891	0.4727
WLS	−0.0009	0.0812	0.0023	0.0531	0.0859	0.4565
LAV	−0.0003	0.0855	0.0037	0.0558	0.1280	0.5657
			$c = 3, n = 100$			
ML	0.0002	0.0312	−0.0010	0.0351	0.0602	0.2521
OLS	−0.0002	0.0635	0.0019	0.0381	0.0407	0.3098
WLS	0.0003	0.0566	0.0012	0.0366	0.0513	0.2940
LAV	0.0002	0.0600	0.0023	0.0386	0.0638	0.3567
			$c = 3, n = 200$			
ML	0.0002	0.0220	−0.0006	0.0249	0.0289	0.1732
OLS	0.0001	0.0449	0.0010	0.0263	0.0221	0.2114
WLS	−0.0002	0.0403	0.0004	0.0259	0.0267	0.2006
LAV	−0.0001	0.0427	0.0010	0.0269	0.0295	0.2320

Note. Samples of size n were drawn from the $BPL(0,1,c)$.
Source: authors' work.

4. Application

In this section, we present two real data examples to demonstrate the flexibility and applicability of the BPL. A total of ten distributions are involved in Monte Carlo simulations.

The models selected for comparison with the BPL are: the beta-normal (BNS, Eugene et al., 2002), bimodal exponential power (BEP, Hassan & Hijazi, 2010), bimodal normal (BN, Hassan & Hijazi, 2010), bimodal Laplace (BL, Harandi & Alamatsaz, 2013), bimodal-symmetric-normal (BSN, Hassan & El-Bassiouni, 2016), bimodal power normal (BPN, Bolfarine et al., 2018), bimodal normal (BND, Vila et al., 2021), compound normal (CN) and compound Laplace (CL). The PDFs of the used models are shown in Table 2.

Table 2. PDFs $f(x;\boldsymbol{\Theta}), x \in R$ of the models used in the fitting data

Model	PDF
BPL	$f(x;a,b,c) = \frac{c}{2b^{c+1}}\lvert x-a\rvert^{c-1}\exp\left[-\left\lvert\frac{x_1-a}{b}\right\rvert^{c}\right]$ $(c \geq 1)$
BEP	$f_{BEP}(x;a,b,c,d) = \frac{c\lvert x-a\rvert^{d}}{2b^{d+1}\Gamma[(d+1)c^{-1}]}\exp\left[-\left\lvert\frac{x-a}{b}\right\rvert^{c}\right]$ $(c \geq 1, d \geq 0)$
BPN	$f_{BPN}(x;a,b,c) = \frac{c2^{c-1}\phi(x;\ a,b)}{b(2^{c}-1)}\Phi\left(\left\lvert\frac{x-a}{b}\right\rvert;0,1\right)^{c-1}$ $(c > 0)$
BN	$f_{BN}(x;a,b) = \frac{2(x-a)^2}{b^3\sqrt{\pi}}\exp\left[-\left(\frac{x-a}{b}\right)^2\right]$
BL	$f_{BL}(x;a,b) = \frac{1}{4b}\left(\frac{x-a}{b}\right)^2\exp\left(-\left\lvert\frac{x-a}{b}\right\rvert\right)$
CN	$f_{CN}(x;a,b,c,d,e) = e\phi(x;a,b) + (1-e)\phi(x;c,d)$ $(c \in R, d > 0)$
BND	$f_{BND}(x;a,b,c) = \frac{\cosh[b^{-1}c(x-a)]}{b\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{x-a}{b}\right)^2-\frac{c^2}{2}\right]$ $(c \in R)$
CL	$f_{CL}(x;a,b,c,d,e) = \frac{e}{2b}\exp\left(-\frac{\lvert x-a\rvert}{b}\right)+\frac{1-e}{2d}\exp\left(-\frac{\lvert x-c\rvert}{d}\right)$ $(c \in R, d > 0)$
BSN	$f_{BSN}(x;a,b,c) = \frac{2b^{1.5}(x-c)^2\exp[-b(x-a)^2]}{\sqrt{\pi}[1+2b(c-a)^2]}$ $(c \in R)$
BNS	$f_{BNS}(x;a,b,c) = \frac{\Gamma(2c)\Phi(x;a,b)^{c-1}}{\Gamma(c)^2}\phi(x;a,b)\big(1-\Phi(x;a;b)\big)^{c-1}$

Note. Symbols ϕ and Φ denote the PDF and CDF of the normal distribution, respectively, $a \in R, b > 0, e \in [0,1]$.
Source: authors' work.

Let be a vector $\widehat{\boldsymbol{\Theta}}$ of the model parameters. The estimation of the model parameters is carried out by the ML method. To avoid local maxima of the logarithmic likelihood function, the optimisation routine is run 100 times with several different starting values that are widely scattered in the parameter space. The Kolmogorov-Smirnov (KS) test (in the Lilliefors variant) was used for model fitting, while the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and Hannan-Quinn Information Criterion (HQIC) were used for model comparisons. Let us note that

$$AIC = -2l + 2p,\ BIC = -2l + p\ln(n),\ HQIC = -2l + 2p\ln\big(\ln(n)\big),$$

where l is the log-likelihood function (56), n is the sample size and p is the number of model parameters.

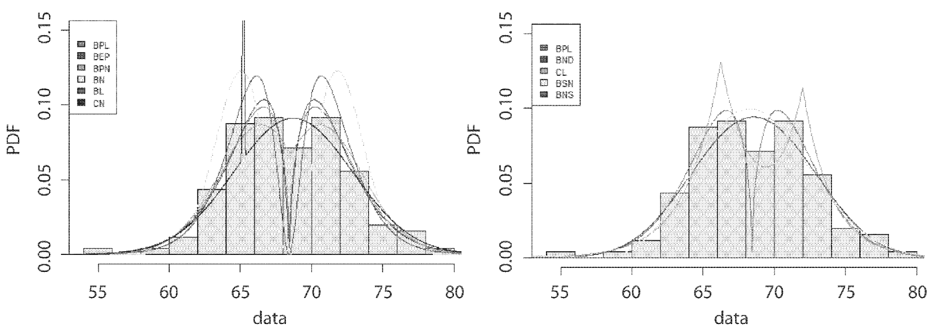
Figures 7–8 show the estimated PDFs for the analysed model. Tables 3–4 display the values of the MLEs and standard errors (in parentheses), the information criteria and the KS test statistics. The lowest values of the information criteria and the KS test statistics are marked in bold.

Let $a, b, \gamma_1, \bar{\gamma}_2$ denote the mean, standard deviation, skewness and excess kurtosis, respectively.

Example 1. The first data set presents the height (in inches) of 126 students from University of Pennsylvania (Cruz-Medina, 2001; Hassan & Hijazi, 2010). The descryptive statistics are: $a = 68.55, b = 4.16, \gamma_1 = -0.05, \bar{\gamma}_2 = 0.05$.

Table 3 shows that the BPN, CL, BPL and BNS models stand out with respect to the KS test statistics. The BPL, CN and BEP models are the best according to the information criteria. Based on the graphical and numerical results, the BPL distribution is considered as one of the best models for the analysed first data set.

Figure 7. Estimated PDF of the analysed distributions (first data set)



Source: authors' work.

Table 3. MLEs (SE), information criteria and KS test statistics (first data set)

Model	MLEs (SE)	AIC	BIC	HQIC	KS
BPL	$\hat{a} = 68.442$ (0.133) $\hat{b} = 3.777$ (0.238) $\hat{c} = 1.492$ (0.102)	**713.780**	**722.289**	**717.237**	0.045
BEP	$\hat{a} = 68.434$ (0.116) $\hat{b} = 2.748$ (1.685) $\hat{c} = 1.231$ (0.400) $\hat{d} = 0.706$ (0.404)	715.366	726.712	719.976	0.050
BPN	$\hat{a} = 68.540$ (0.272) $\hat{b} = 3.694$ (0.298) $\hat{c} = 2.171$ (0.681)	718.808	727.317	722.265	**0.040**
BN	$\hat{a} = 68.440$ (0.068) $\hat{b} = 3.382$ (0.123)	794.392	800.060	796.697	0.118
BL	$\hat{a} = 68.428$ (0.072) $\hat{b} = 1.135$ (0.058)	726.825	732.497	729.120	0.060
CN	$\hat{a} = 65.217$(0.026) $\hat{b} = 0.050$ (0.017) $\hat{c} = 68.715$ (0.384) $\hat{d} = 4.175$ (0.270) $\hat{e} = 0.048$ (0.022)	714.167	728.348	719.928	0.047

Table 3. MLEs (SE), information criteria and KS test statistics (first data set; cont.)

Model	MLEs (SE)	AIC	BIC	HQIC	KS
BND	$\hat{a} = 68.550\ (0.371)$ $\hat{b} = 3.581\ (1.797)$ $\hat{c} = 0.581\ (1.150)$	721.662	730.170	725.118	0.051
CL	$\hat{a} = 66.250\ (0.025)$ $\hat{b} = 2.386\ (0.353)$ $\hat{c} = 72.000\ (0.020)$ $\hat{d} = 1.958\ (0.384)$ $\hat{e} = 0.600\ (0.065)$	718.662	732.840	724.423	0.042
BSN	$\hat{a} = 64.136\ (0.676)$ $\hat{b} = 57.338\ (0.728)$ $\hat{c} = 0.022\ (0.003)$	719.728	728.236	723.184	0.058
BNS	$\hat{a} = 68.596\ (0.352)$ $\hat{b} = 1.480\ (0.931)$ $\hat{c} = 0.186\ (0.196)$	721.187	729.696	724.644	0.046

Note. The best results are marked in bold.
Source: authors' work.

Example 2. The second data set presents 500 units observed for variable Z = b.weight, which is the ultrasound weight (fetal weight in grams). These data are available for downloading at http://www.mat.uda.cl/hsalinas/data/weight.rar. The descriptive statistics are: $a = 3210.36, b = 834.09, \gamma_1 = 0.07, \bar{\gamma}_2 = -0.93$.

Table 4. MLEs (SE), information criteria and KS test statistics (second data set)

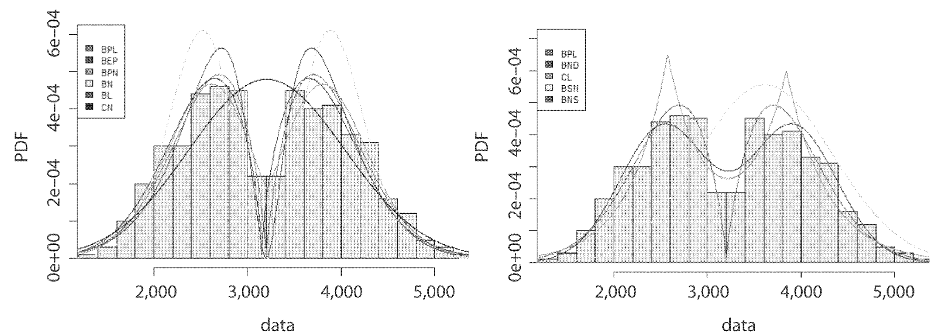
Model	MLEs (SE)	AIC	BIC	HQIC	KS
BPL	$\hat{a} = 3201.578\ (9.003)$ $\hat{b} = 808.646\ (21.590)$ $\hat{c} = 1.760\ (0.063)$	**8086.885**	**8099.528**	**8091.846**	0.023
BEP	$\hat{a} = 3151.489\ (7.324\ )$ $\hat{b} = 903.144\ (163.642)$ $\hat{c} = 1.945\ (0.384\ )$ $\hat{d} = 0.622\ (0.173)$	8094.724	8111.582	8101.339	0.043
BPN	$\hat{a} = 3209.213\ (21.069)$ $\hat{b} = 660.716\ (21.564)$ $\hat{c} = 3.747\ (0.353)$	8088.208	8100.850	8093.170	**0.019**
BN	$\hat{a} = 3201.644\ (5.621)$ $\hat{b} = 680.390\ (12.422)$	8238.393	8246.822	8241.701	0.078
BL	$\hat{a} = 3200.725\ (5.812)$ $\hat{b} = 240.437\ (6.208)$	8124.837	8133.266	8128.145	0.034
CN	$\hat{a} = 3297.540\ (224.593)$ $\hat{b} = 831.199\ (37.564)$ $\hat{c} = 3003.800$ $\hat{d} = 799.626\ (15.790)$ $\hat{e} = 0.702\ (0.742)$	8154.037	8175.110	8162.300	0.066
BND	$\hat{a} = 3223.212\ (27.591)$ $\hat{b} = 466.244\ (20.094)$ $\hat{c} = 1.481\ (0.090)$	8094.226	8106.870	8099.187	0.029

Table 4. MLEs (SE), information criteria and KS test statistics (second data set; cont.)

Model	MLEs (SE)	AIC	BIC	HQIC	KS
CL	$\hat{a} = 3842.806$ $\hat{b} = 437.147\ (30.587)$ $\hat{c} = 2575.420$ $\hat{d} = 404.305\ (29.343)$ $\hat{e} = 0.500\ (0.024)$	8135.481	8156.554	8143.750	0.028
BSN	$\hat{a} = 3200.089$ $\hat{b} = 703.798$ $\hat{c} = 8.402\text{e} - 07$	8363.613	8376.257	8368.575	0.272
BNS	$\hat{a} = 3207.324$ $\hat{b} = 792.528$ $\hat{c} = 0.912$	8149.743	8162.387	8154.705	0.065

Note. The best results are marked in bold.
Source: authors' work.

Figure 8. Estimated PDF of the analysed distributions (second data set)



Source: authors' work.

Table 4 shows that the BPN, BPL, CL and BND models stand out with respect to the KS test statistics. The BPL, BPN, BEP and BND models are the best according to the information criteria. Based on the graphical and the numerical results, the BPL distribution is considered as one of the best models for the analysed second data set.

5. Conclusions

The authors of this paper have defined the bimodal power Laplace distribution from the standard Laplace distribution. The PDF of the analysed distribution can also be obtained by applying the Fernández-Steel transformation (Fernández & Steel, 1998) to the Weibull distribution with a scale parameter equal to 1. The main goal of this paper was to present the unknown properties of the distribution in question. It turned out that the

PDF of the bimodal power Laplace distribution is the same as the PDF of the double Weibull distribution (Balakrishnan & Kocherlakota, 1985).

The goals of the article, which included a brief summary of the known and presentation of the unknown properties of the BPL distribution, and a chronological overview of the distributions belonging to the large family of Laplace distributions and four estimation methods, have been achieved.

Simulation examples showed that estimation procedures performed well. Real data examples demonstrate that the bimodal power Laplace distribution is a flexible model that deserves to be added to the existing distributions for modelling bimodal data.

Future research directions could involve extending this model to handle multimodal datasets with more than two modes, and developing applications in emerging fields like machine learning and data science.

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Appendix

A. Proof of Theorem 2. Let $U \sim BPL(0, 1, c), c \geq 1$. Using (5), we have

$$f(u; c) = \frac{c}{2}|u|^{c-1}\exp(-|u|^c).$$

Let $u < 0$, then

$$f'(u; c) = f(u; c)(-u)^{-1}[1 - c + c(-u)^c]. \tag{A1}$$

Let us assume that $f'(u; c) = 0 \Rightarrow 1 - c + c(-u)^c = 0$. Solving this equation, we obtain

$$u = -\left(1 - \frac{1}{c}\right)^{1/c}.$$

There is a maximum at point u, because $f''(u; c) < 0$. If $X = a + bU$, then we have

$$x_{m1} = a - b\left(1 - \frac{1}{c}\right)^{1/c}.$$

Similarly, for $u \geq 0$, we have

$$x_{m2} = a + b\left(1 - \frac{1}{c}\right)^{1/c}.$$

The proof is complete.

B. Proof of Theorem 3. Let $U \sim BPL(0, 1, c), c \geq 1$. Let $u < 0$, then using (A1), we have

$$f''^{(u)} = \frac{-c(-u)^c e^{-(-u)^c}[3c(-u)^c - 3c + c^2 - 3c^2(-u)^c + c^2(-u)^{2c} + 2]}{2u^3}.$$

Solving equation $f''(u) = 0$, we obtain

$$c^2t^2 + (3c - 3c^2)t + c^2 - 3c + 2 = 0 \quad (t = (-u)^c). \tag{A2}$$

Solving (A2), we have

$$t = \frac{3c - 3 \mp \sqrt{(c-1)(5c-1)}}{2c}$$

and

$$u = -\left(\frac{3c - 3 \mp \sqrt{(c-1)(5c-1)}}{2c}\right)^{1/c} \quad (c > 2). \tag{A3}$$

If $X_1 = a + bU$, then we have

$$x_{\mathrm{I,II}} = a - b\left(\frac{3c - 3 \mp \sqrt{(c-1)(5c-1)}}{2c}\right)^{\frac{1}{c}}.$$

Similarly, for $u \geq 0$, we obtain

$$x_{\mathrm{III,IV}} = a + b\left(\frac{3c - 3 \mp \sqrt{(c-1)(5c-1)}}{2c}\right)^{\frac{1}{c}}.$$

The proof is complete.

C. Proof of Theorem 5. Let $\tilde{X} \sim BPL(0, b, c)$. The non-central moments of $\tilde{X}$ can be written as

$$E\left(\tilde{X}^k\right) = I_1 + I_2, \tag{A4}$$

where

$$I_1 = \frac{c}{2b}\int_{-\infty}^{0} x^k \left(\frac{-x}{b}\right)^{c-1} \exp\left[-\left(\frac{-x}{b}\right)^c\right] dx, \tag{A5}$$

$$I_2 = \frac{c}{2b}\int_{-\infty}^{0} x^k \left(\frac{x}{b}\right)^{c-1} \exp\left[-\left(\frac{x}{b}\right)^c\right] dx. \tag{A6}$$

If $\left(\frac{-x}{b}\right)^c = t$, then $\frac{-c}{b}\left(\frac{-x}{b}\right)^{c-1} dx = dt$. As a result of simple transformations, we can write (A5) as

$$I_1 = 0.5\int_{-\infty}^{0} \left[-bt^{\frac{1}{c}}\right]^k e^{-t} dt = 0.5(-b)^k \int_{-\infty}^{0} t^{k/c} e^{-t} dt$$

Using the definition of the gamma function, we have

$$I_1 = 0.5(-b)^k \Gamma\left(\frac{k}{c} + 1\right). \tag{A7}$$

If $0.5\left(\frac{x}{b}\right)^c = t$, then $\frac{c}{b}\left(\frac{x}{b}\right)^{c-1} dx = dt$. As a result of simple transformations, we can write (A6) as

$$I_2 = 0.5b^k\Gamma\left(\frac{k}{c}+1\right). \tag{A8}$$

Thus, (A4) is given by

$$E\left(\tilde{X}_1^k\right) = I_1 + I_2 = 0.5\Gamma\left(\frac{k}{c}+1\right)[(-b)^k + b^k].$$

The proof is complete.

D. R codes

```
dBPL = function(x, a, b, c) {
 z1=abs((x-a)/b)^c; z2=abs((x-a)/b)^(c-1)
 return(1/2/b*c*z2*exp(-z1)) }
pBPL = function(x, a, b, c) {
 return(ifelse(x<a,0.5*exp(-((a-x)/b)^c),1-0.5*exp(-((x-a)/b)^c))) }
qBPL = function(p, a, b, c) {
 if ((p<0.5) & (p>=0)) res=a-b*(-log(2*p))^(1/c)
 if ((p>=0.5) & (p<=1)) res=a+b*(-log(2-2*p))^(1/c)
 return(res) }
rBPL=function(n, a, b, c) {
 x=numeric(n)
 for (i in 1:n) {
 U=runif(1,0,1)
 x[i]=ifelse(U<0.5,a-b*(-log(2*U))^(1/c),a+b*(-log(2-2*U))^(1/c)) }
 return(x) }
```