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Goodness-of-fit testing for normality where alternative distributions have undefined or constant skewness and excess kurtosis

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Abstract. The aim of the paper is to examine goodness-of-fit testing for normality where alternative distributions have undefined or constant skewness and excess kurtosis. The first step involves collecting a set of normality-oriented goodness-of-fit tests (GoFTs) recommended for use in the source literature. The second step is to create a family of distributions with undefined or constant skewness and excess kurtosis. For greater clarity, the alternative distributions have been divided into three groups. Group I consisted of symmetric alternatives with undefined excess kurtosis, group II of symmetric alternatives with constant non-zero excess kurtosis, and group III comprised asymmetric alternatives with constants non-zero skewness and non-zero excess kurtosis. An appropriate similarity measure (SM) of a given alternative distribution to the normal distribution was applied. The third step involved a Monte Carlo simulation.

The obtained results indicate that the GoFT power for the analysed alternatives and for a given sample size remain relatively unchanged despite the significant decrease in SM. Several GoFTs proved incapable of capturing differences between normal distribution and the alternatives. This situation occurs even when SM equals 0.15. If SM equals 1, then the probability density functions (PDFs) are identical. The best tests for group I are the Gel-Miao-Gastwirth (SJ) and Robust Jarque-Bera (RJB) tests (positive excess kurtosis) as well as the Coin and Chen-Shapiro (CS) tests (negative excess kurtosis), for group II the SJ, 1st Hosking and RJB tests, while the Zhang-Wu (ZA) and CS tests are best for group III.

Keywords: normal distribution, goodness-of-fit testing, excess kurtosis modelling
JEL: C12, C13, C15

Testowanie normalności rozkładów alternatywnych z niezdefiniowanymi lub stałymi skośnością i ekscesem

Streszczenie. Celem artykułu jest przetestowanie normalności rozkładów z niezdefiniowanymi lub stałymi skośnością i ekscesem. Pierwszym krokiem do realizacji tego celu jest przygotowanie zbioru testów zgodności (TZ) dla normalności zalecanych w literaturze. Drugi krok polega na utworzeniu rodziny rozkładów o niezdefiniowanych lub stałych skośności i ekscesie. Dla większej przejrzystości rozkłady alternatywne podzielono na trzy grupy: do grupy I przypisano alternatywy symetryczne o niezdefiniowanym ekscesie, do grupy II – alternatywy symetryczne o stałym niezerowym ekscesie, do grupy III – alternatywy asymetryczne o stałych niezerowych skośności i ekscesie. Użyto odpowiedniej miary podobieństwa (MP) danej alternatywy do rozkładu normalnego. Trzeci krok to symulacja Monte Carlo.

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Uzyskane wyniki pokazują, że moc TZ dla analizowanych alternatyw i dla danej liczebności próby pozostaje na relatywnie niezmienionym poziomie pomimo znacznego obniżenia wartości MP. Wiele TZ okazało się niezdolnych do uchwycenia różnic między rozkładami normalnym a alternatywnym, nawet gdy MP wynosi 0,15. Jeżeli MP wynosi 1, to przebiegi funkcji gęstości są identyczne. Najlepszymi testami są: dla grupy I – testy Gel-Miao-Gastwirtha (SJ) i robust Jarque'a-Bery (RJB; eksces dodatni) oraz testy Coina i Chen-Shapira (CS; eksces ujemny), dla grupy II – testy SJ, 1. test Hoskinga i RJB, a dla grupy III – testy Zhanga-Wu (ZA) i CS.
Słowa kluczowe: rozkład normalny, testy zgodności, modelowanie ekscesu

1. Introduction

One cannot disagree with the statement that the most common package of statistical inference procedures is applied to normal distribution parameters. However, before analysts use one of these procedures, they need to check whether the empirical data come from the general population in which normal distribution is 'in force'. In other words, analysts are required to perform a goodness-of-fit test (GoFT) for normality.

Many GoFTs are discussed in the statistical literature. The most common normality test procedures available in statistical software are: the KS test (Kolmogorov, 1933; Smirnov, 1948), the LF test (Lilliefors, 1967), the CVM test (Cramér, 1928), the AD test (Anderson & Darling, 1952) and the SW test (Shapiro & Wilk, 1965). The PoweR package (Lafaye de Micheaux & Tran, 2016) from the R software is noteworthy in this case, as it provides a large set of generators of pseudo-random numbers that follow probability distributions used both frequently and sporadically. Moreover, the package offers many GoFTs for normality, uniformity and laplacity (see Section 4).

Recently, many articles have been devoted to GoFTs for normality. They include Kellner and Celisse (2019) and several others mentioned in Table 1 (see Section 2). Many of these papers define and compare GoFTs using a big family of alternatives. This family can be divided into four sets, depending on the support (see e.g. Uyanto, 2022). The family of alternatives can also be divided into three sets. These are symmetric distributions with undefined excess kurtosis $\bar{\gamma}_2$, symmetric distributions with $\bar{\gamma}_2 \neq 0$ and asymmetric distributions with constant skewness $\gamma_1 \neq 0$ and $\bar{\gamma}_2 \neq 0$.

The aim of the paper is to study GoFT testing for normality where alternative distributions have undefined or constant skewness γ_1 and excess kurtosis $\bar{\gamma}_2$.

This paper is also devoted to GoFTs for normality. Its purpose, however, is not to define a new test, but to calculate GoFT powers for various alternatives with undefined or constant γ_1 and $\bar{\gamma}_2$ (see Section 3.1). In the paper, the scale parameter of alternatives is non-constant and is the argument of the similarity measure (SM) of the alternative distribution to normal distribution (see formula (1)). The values of the scale parameter are chosen to obtain appropriate SM values, including the maximum SM value (see Section 4). If SM equals 1, then the normal distribution is a special case of the alternative distribution.

Will the GoFT power for a given alternative distribution and sample size change when SM is equal to 0.75, 0.50 and 0.25? Which tests are recommended, and which are not for the analysed alternatives?

To answer these questions, it is necessary to create a set of recently recommended GoFTs and groups of alternatives I–III with undefined or constant γ_1 and $\overline{\gamma}_2$, then define the SM of the alternative distribution to the normal distribution and compare the power of the recommended GoFTs for the analysed alternatives, sample size and SM values.

2. Goodness-of-fit tests for the Monte Carlo simulation

Hypothesis H_0 states: data come from normal distribution. Hypothesis H_1 obviously negates H_0.

Table 1 presents the recently recommended and sorted by year GoFTs for normality ($n \leq 100$) when alternatives are symmetric and asymmetric, respectively. These GoFTs are used in the Monte Carlo simulations (see Section 4).

Table 1. GoFTs for normality recommended when alternatives are symmetric or asymmetric, $n \leq 100$

GoFT	Alternatives S – symmetric A – asymmetric	Recommended by
Anderson-Darling test (AD) (Anderson & Darling, 1952)	S	Wijekularathna et al. (2022), Yap and Sim (2011)
	A	Afeez et al. (2018), Khatun (2021), Yap and Sim (2011)
Shapiro-Wilk test (SW) (Shapiro & Wilk, 1965)	S	Desgagné et al. (2022), Mbah and Paothong (2015), Mishra et al. (2019), Nosakhare and Bright (2017), Wijekularathna et al. (2022), Yap and Sim (2011)
	A	Afeez et al. (2018), Bayoud (2021), Coin (2008), Hernandez (2021), Khatun (2021), Mishra et al. (2019), Romão et al. (2010), Wijekularathna et al. (2022), Yap and Sim (2011)
Kurtosis test (KT) (Shapiro et al., 1968)	S	Mishra et al. (2019)
	A	
D'Agostino skewness test (AS) (D'Agostino, 1971)	S	
	A	
Shapiro-Francia test (SF) (Shapiro & Francia, 1972)	S	Nosakhare and Bright (2017)
	A	Khatun (2021), Nosakhare and Bright (2017)
D'Agostino-Pearson test (AP) (D'Agostino & Pearson, 1973)	S	Mishra et al. (2019), Wijekularathna et al. (2022), Yap and Sim (2011)
	A	Mishra et al. (2019)
Ryan-Joiner test (RJ) (Ryan & Joiner, 1976)	S	Wijekularathna et al.(2022)
	A	Nosakhare and Bright (2017)
T_{1n} test (T_{1n}) (LaRiccia, 1986)	S	.
	A	Torabi et al. (2016)
Jarque-Bera test (JB) (Jarque & Bera, 1987)	S	Afeez et al. (2018), Brys et al. (2008), Mbah and Paothong (2015), Yap and Sim (2011)
	A	Brys et al. (2008), Yazici and Yolacan (2007)
1st Hosking test (H1) (Hosking, 1990)	S	Arnastauskaitė et al. (2021)
	A	
1st Cabaña-Cabaña test (CC) (Cabaña & Cabaña, 1994)	S	.
	A	Uyanto (2022)

Table 1. GoFTs for normality recommended when alternatives are symmetric or asymmetric, $n \leq 100$ (cont.)

GoFT	Alternatives S – symmetric A – asymmetric	Recommended by
Chen-Shapiro test (CS) (Chen et al., 1995)	S	Desgagné et al. (2022), Romão et al. (2010)
	A	Romão et al. (2010)
Adjusted Jarque-Bera test (AJB) (Urzua, 1996)	S	Wijekularathna et al. (2022)
	A	Nosakhare and Bright (2017)
Bonett-Seier test (BS) (Bonett & Seier, 2002)	S	Romão et al. (2010)
	A	.
ZA Zhang-Wu test (ZA) (Zhang & Wu, 2005)	S	Uhm and Yi (2023)
	A	Romão et al. (2010), Sulewski (2019), Uhm and Yi (2023), Uyanto (2022)
ZC Zhang-Wu test (ZC) (Zhang & Wu, 2005)	S	Uhm and Yi (2023)
	A	Romão et al. (2010), Uhm and Yi (2023)
Gel-Miao-Gastwirth test (SJ) (Gel at el., 2007)	S	Romão et al. (2010), Sulewski (2019), Torabi et al. (2016), Uyanto (2022)
	A	.
β_3^2 Coin test (β_3^2) (Coin, 2008)	S	Coin (2008), Romão et al. (2010)
	A	Coin (2008)
Robust Jarque-Bera test (RJB) (Gel & Gastwirth, 2008)	S	Bayoud (2021), Sulewski (2019), Torabi et al. (2016), Uyanto (2022)
	A	.
H_n test (H_n) (Torabi et al., 2016)	S	.
	A	Torabi et al. (2016)
X_{APD} test (X_{APD}) (Desgagné & Lafaye de Micheaux, 2018)	S	Desgagné et al. (2022), Wijekularathna et al. (2022)
	A	Desgagné et al. (2022)
Z_{EPD} test (Z_{EPD}) (Desgagné & Lafaye de Micheaux, 2018)	S	Wijekularathna et al. (2022)
	A	.
B_v test (B_v) (Tavakoli et al., 2019)	S	.
	A	Tavakoli et al. (2019)
Delta test (δ) (Bayoud, 2021)	S	.
	A	Bayoud (2021)
Modified Lilliefors test ($LF_{\overline{\alpha},\overline{\beta}}$) (Sulewski, 2021b)	S	Sulewski (2021b, 2022)
	A	.

Note. The GoFTs have been ordered chronologically. '.' means there was no recommendation for a given GoFT with the particular type of alternatives (symmetric or asymmetric).

Source: author's work based on a literature review.

3. Alternative distributions and similarity measure

3.1. Alternative distributions

We focus on unimodal, bimodal and multimodal distributions with undefined or constant skewness γ_1 and excess kurtosis $\bar{\gamma}_2$ supported on the whole real axis, on a bounded interval and on a semi-infinite interval. The family of alternatives consists of 24 distributions, the PDFs of which are shown in Tables 2–4. These are:

- group I – symmetric distributions with undefined $\bar{\gamma}_2$: Cauchy (C) and slash (SL);
- group II – symmetric distributions with constant $\bar{\gamma}_2 \neq 0$: arcsine (AS), bimodal Laplace (BL), bimodal Normal (BN), cosine (COS), hyperbolic secant (HS), inverted U-shaped parabolic (IUP), Laplace (L) or double exponential, logistic (LOG) or Tukey ($\lambda = 0$), raised cosine (RCOS), semicircle (SC), sine (SIN), triangular (TRI), uniform (U), U-shaped parabolic (UP) or U-quadratic and V-shaped (V);
- group III – asymmetric distributions with constants $\gamma_1 \neq 0, \bar{\gamma}_2 \neq 0$: exponential (EXP), Gumbel (GU), half-logistic (HL), half-Normal (HN), log-Weibull (LW) or extreme-value, Maxwell (MX) and Rayleigh (RY).

The family of alternatives consists of: 18 unimodal distributions, 2 bimodal (BL, BN), 3 anti-modal (AS, UP, V) and one multimodal (U) distributions; 5 distributions supported on $[0, \infty)$ (EXP, HL, HN, MX, RY), 10 distributions supported on a finite interval (AS, COS, IUP, RCOS, SC, SIN, TRI, U, UP, V), and 9 distributions supported on an infinite interval (BL, BN, C, GU, HS, L, LOG, LW, SL).

Table 2. Symmetric alternatives supported on R with undefined $\bar{\gamma}_2 = 0$, zero-position parameter and scale parameter $a > 0$, forming group I

Distribution	PDF
C	$f_C(x; a) = \dfrac{a}{\pi(a^2 + x^2)}$
SL	$f_{SL}(x) = \begin{cases} \left(\varphi(0; 0,1) - \varphi(x; 0,1)\right)x^{-2} & x \neq 0 \\ \left(2\sqrt{2\pi}\right)^{-1} & x = 0 \end{cases}$

Source: author's work based on a literature review.

Table 3. Symmetric distributions with constant $\bar{\gamma}_2 \neq 0$, zero-position parameter and scale parameter $a > 0$, forming group II

Distribution	PDF	Support	$\bar{\gamma}_2$
AS	$f_{AS}(x; a) = \left(\pi\sqrt{a^2 - x^2}\right)^{-1} \quad (a > 0)$	$[-a, a]$	-1.5
BL	$f_{BL}(x; a) = \dfrac{\lvert x\rvert}{2a^2} exp\left(-\dfrac{\lvert x\rvert}{a}\right)$	R	$\dfrac{1}{3}$
BN	$f_{BN}(x; a) = \dfrac{2x^2}{a^3\sqrt{\pi}} exp\left(-\dfrac{x^2}{a^2}\right)$	R	$-\dfrac{4}{3}$
COS	$f_{COS}(x; a) = \dfrac{1}{2a} cos\left(\dfrac{x}{a}\right)$	$\left[-\dfrac{a\pi}{2}, \dfrac{a\pi}{2}\right]$	$\dfrac{2(96 - \pi^4)}{(\pi^2 - 8)^2}$
HS	$f_{HS}(x; a) = \dfrac{1}{a\pi} sech\left(\dfrac{x}{a}\right)$	R	2
IUP	$f_{IUP}(x; a) = \dfrac{3(a^2 - x^2)}{4a^3}$	$[-a, a]$	$-\dfrac{6}{7}$
L	$f_L(x; a) = \dfrac{1}{2a} exp\left(-\dfrac{\lvert x\rvert}{a}\right)$	R	3
LOG	$f_{LOG}(x; a) = \dfrac{1}{4a} sech^2\left(\dfrac{x}{2a}\right)$	R	1.2

Table 3. Symmetric distributions with constant $\bar{\gamma}_2 \neq 0$, zero-position parameter and scale parameter $a > 0$, forming group II (cont.)

Distribution	PDF	Support	$\bar{\gamma}_2$
RCOS	$f_{RCOS}(x;a) = \frac{1}{2a}\left[1 + cos\left(\frac{\pi x}{a}\right)\right]$	$[-a,a]$	$\frac{6(96-\pi^4)}{5(\pi^2-6)^2}$
SC	$f_{SC}(x;a) = \frac{2\sqrt{a^2-x^2}}{\pi a^2}$	$[-a,a]$	-1
SIN	$f_{SIN}(x;a) = \frac{\pi}{2a} sin\left(\pi\frac{2x+a}{2a}\right)$	$\left[-\frac{a}{2},\frac{a}{2}\right]$	$\frac{2(96-\pi^4)}{(\pi^2-8)^2}$
TRI	$f_{TRI}(x;a) = \frac{a-\lvert x\rvert}{a^2}$	$[-a,a]$	-0.6
U	$f_U(x;a) = (2a)^{-1}$	$[-a,a]$	-1.2
UP	$f_{UP}(x;a) = \frac{3x^2}{2a^3}$	$[-a,a]$	$-\frac{38}{21}$
V	$f_V(x;a) = \begin{cases} -xa^{-2} & -a \le x \le 0 \\ xa^{-2} & 0 < x \le a \end{cases}$	$[-a,a]$	$-\frac{5}{3}$

Source: author's work based on a literature review.

Table 4. Asymmetric distributions with constants $\gamma_1 \neq 0, \bar{\gamma}_2 \neq 0$ and scale parameter $a > 0$, forming group III

Distribution	PDF	Support	γ_1	$\bar{\gamma}_2$
EXP	$f_{EXP}(x;a) = \frac{1}{a} exp\left(-\frac{x}{a}\right)$	$[0,\infty)$	2	6
GU	$f_{GU}(x;a) = \frac{1}{a} exp\left[-exp\left(-\frac{x}{a}\right) - \frac{x}{a}\right]^*$	R	1.140*	2.4
HL	$f_{HL}(x;a) = \frac{1}{2a}\operatorname{sech}^2\left(\frac{x}{2a}\right)$	$[0,\infty)$	1.540*	3.584*
HN	$f_{HN}(x;a) = \frac{\sqrt{2}}{a\sqrt{\pi}}\exp\left(\frac{-x^2}{2a^2}\right)$	$[0,\infty)$	$\frac{\sqrt{2}(4-\pi)}{(\pi-2)^{1.5}}$	$\frac{8(\pi-3)}{(\pi-2)^2}$
LW	$f_{LW}(x;a) = \frac{1}{a} exp\left[\frac{x}{a} - exp\left(\frac{x}{a}\right)\right]^{**}$	R	−1.140*	2.4
MX	$f_{MX}(x;a) = \frac{x^2\sqrt{2}}{a^3\sqrt{\pi}}\exp\left(\frac{-x^2}{2a^2}\right)$	$[0,\infty)$	$\frac{2\sqrt{2}(16-5\pi)}{(3\pi-8)^{1.5}}$	$\frac{-4(3\pi^2-40\pi+96)}{(3\pi-8)^2}$
RY	$f_{RY}(x;a) = \frac{x}{a^2}\exp\left(\frac{-x^2}{2a^2}\right)$	$[0,\infty)$	$\frac{2\sqrt{\pi}(\pi-3)}{(4-\pi)^{1.5}}$	$\frac{-2(3\pi^2-12\pi+8)}{(4-\pi)^2}$

Note. *denotes approximate value, ** denotes $f_{LW}(-x;a) = f_{GU}(x;a)$.
Source: author's work based on a literature review.

3.2. Similarity measure

Let $f(x;a)$ be a PDF of the alternative distribution with scale parameter a. The SM of alternative distribution f to the normal distribution is defined as (Sulewski, 2022)

$$M(a;\mu,\sigma) = \int_{-\infty}^{\infty} min[f(x;a),\phi(x;\mu,\sigma)]dx,$$

where $\phi(x;\mu,\sigma)$ is the PDF of the normal distribution, $M(a;\mu,\sigma)$ takes the value of $[0,1]$, $M(a;\mu,\sigma) = 1$ when the PDFs are identical.

More details on distance and SM can be found in Sulewski (2021a).

The Figure in the Appendix shows the PDF of the normal distribution and the alternatives applied in the Monte Carlo simulation. The values of the SM are provided in the respective keys to the Figure. If a increases (decreases) for AS, BL, BN, C, COS, EXP, HN, IUP, L, LOG, RY (GU, HL, HS, LW, MX, RCOS, SC, SIN, SL, TRI, U, UP, V), then the SM decreases.

4. Monte Carlo simulations

The values of scale parameter a of the alternatives were selected to obtain appropriate SM values, including the maximum M value.

For a given alternative distribution, 20 (symmetric alternatives) or 21 (asymmetric alternatives) large-scale experiments were performed, each dedicated to one of the GoFTs. Each experiment involved generating 10^4 samples of sizes $n = 25$ or $n = 50$. The samples followed the alternative distribution. Each sample was tested for normality at the $\alpha = 0.05$ significance level.

All calculations were performed in the R software using the functions presented in Table 6. A research tool facilitating the Monte-Carlo-power-simulation studies for GoFTs in R, called the PowerR package (Lafaye de Micheaux & Tran, 2016), is very helpful in the process – function 'statcompute()' calculates the test statistic value and the p-value for the GoFT described by the 'stat.index' argument, the sample described by the 'data' argument, and the significance level described by the 'level' argument. For example, the function for the ZA test reads: statcompute(stat.index = 4, data = sample, significance level = 0.05). For more information, see Table 5.

Table 5. The functions of the used GoFTs

GoFT	R codes	GoFT	R codes
AD	ad.test	BS	bonett.test
SW	shapiro.test	ZA	statcompute(stat.index = 4...)
KT	kurtosis.norm.test	ZC	statcompute(stat.index = 3...)
AS	agostino.test	SJ	sj.test
SF	sf.test	β_3^2	statcompute(stat.index = 30...)
AP	dagoTest	RJB	rjb.test
RJ	own function, see Appendix	H_n	own function, see Appendix
T_{1n}	own function, see Appendix	X_{APD}	statcompute(stat.index = 36...)
JB	jarque.test	Z_{EPD}	statcompute(stat.index = 37...)
H1	statcompute(stat.index = 10...)	B_v	own function, see Appendix
CC	statcompute(stat.index = 19...)	δ	own function, see Appendix
CS	statcompute(stat.index = 26...)	$LF_{\overline{\alpha},\overline{\beta}}$	own function, see Appendix
AJB	ajb.norm.test		

Source: author's work based on a literature review.

Table 6 shows the critical values (cv) calculated as 10^6 ordered statistics values and type I error levels for the GoFTs not implemented in the R software when sample sizes $n = 25$ or $n = 50$ and significance level $\alpha = 0.05$.

Table 6. Critical values and type I error levels for GoFTs, $\alpha = 0.05$

GoFT	$n = 25$		$n = 50$	
	cv	type I error	*cv*	type I error
RJ	0.9932	0.051	0.9957	0.053
T_{1n}	4.0234	0.051	3.9328	0.050
δ	0.1102	0.049	0.0788	0.050
$LF_{0,0}$	0.1494	0.050	0.1132	0.050
$LF_{0,1}$	0.1614	0.048	0.1176	0.050
$LF_{1,0}$	0.1614	0.050	0.1176	0.049
$LF_{1,1}$	0.1588	0.049	0.1167	0.049
H_n	0.0006	0.058	0.0003	0.054
B_v	0.1175	0.049	0.1017	0.050

Source: author's work.

5. Symmetric distributions with undefined excess kurtosis

Table 7 shows GoFT powers for group I of alternatives. The top three GoFTs by average power are marked in bold.

Table 7. GoFT powers for group I of alternatives

Specification	$n = 25$				$n = 50$			
	$C(a), \overline{\gamma}_2$ – undefined							
a	0.732	0.996	2.257	5.998	0.732	0.996	2.257	5.998
$M(a; 0, 1)$	0.810	0.750	0.500	0.250	0.810	0.750	0.500	0.250
AD	0.937	0.934	0.937	0.932	0.997	0.997	0.997	0.998
SW	0.925	0.925	0.925	0.924	0.996	0.997	0.996	0.997
KT	0.922	0.921	0.922	0.923	0.996	0.996	0.995	0.997
AS	0.816	0.817	0.816	0.817	0.909	0.905	0.904	0.909
SF	0.941	0.940	0.941	0.940	0.997	0.998	0.997	0.998
AP	0.905	0.905	0.905	0.904	0.992	0.992	0.993	0.992
RJ	0.939	0.939	0.939	0.939	0.997	0.998	0.997	0.998
JB	0.918	0.918	0.918	0.918	0.995	0.996	0.995	0.996
H1	**0.947**	**0.948**	**0.947**	**0.949**	**0.998**	**0.999**	**0.998**	**0.999**
CS	0.920	0.917	0.920	0.918	0.995	0.995	0.995	0.996
AJB	0.925	0.925	0.925	0.926	0.995	0.996	0.995	0.996
BS	0.932	0.929	0.932	0.931	0.998	0.998	0.997	0.999
ZA	0.918	0.918	0.918	0.917	0.994	0.994	0.994	0.995
ZC	0.906	0.906	0.906	0.905	0.992	0.994	0.993	0.994
SJ	**0.958**	**0.958**	**0.958**	**0.960**	**0.999**	**0.999**	**0.999**	**0.999**
β_3^2	0.928	0.928	0.928	0.931	0.997	0.998	0.996	0.998
RJB	**0.947**	**0.948**	**0.947**	**0.948**	**0.997**	**0.998**	**0.998**	**0.999**
X_{APD}	0.939	0.941	0.939	0.940	0.997	0.998	0.997	0.999
Z_{EPD}	0.929	0.930	0.929	0.930	0.997	0.998	0.997	0.998
$LF_{1,1}$	0.920	0.921	0.920	0.917	0.994	0.994	0.995	0.995

Table 7. GoFT powers for group I of alternatives (cont.)

Specification	$n = 25$				$n = 50$			
				SL, $\bar{\gamma}_2$ **– undefined**				
b	2.063	1.372	0.670	0.262	2.063	1.372	0.670	0.262
$M(;0,b)$	0.848	0.750	0.500	0.250	0.848	0.750	0.500	0.250
AD	0.901	0.910	0.910	0.907	0.993	0.992	0.992	0.993
SW	0.895	0.905	0.905	0.903	0.993	0.991	0.990	0.993
KT	0.902	0.908	0.908	0.909	0.995	0.993	0.993	0.994
AS	0.802	0.810	0.810	0.809	0.903	0.900	0.903	0.912
SF	0.917	0.924	0.924	0.922	0.995	0.994	0.993	0.995
AP	0.886	0.894	0.894	0.892	0.990	0.986	0.986	0.988
RJ	0.914	0.922	0.922	0.921	0.995	0.994	0.993	0.994
JB	0.900	0.907	0.907	0.908	0.993	0.991	0.990	0.992
H1	**0.920**	**0.926**	**0.926**	**0.922**	**0.996**	**0.995**	**0.994**	**0.996**
CS	0.889	0.900	0.900	0.899	0.992	0.989	0.989	0.991
AJB	0.907	0.914	0.914	0.914	0.994	0.992	0.992	0.993
BS	0.901	0.904	0.904	0.905	0.996	0.995	0.994	0.995
ZA	0.890	0.900	0.900	0.900	0.991	0.989	0.988	0.990
ZC	0.881	0.890	0.890	0.888	0.990	0.986	0.986	0.988
SJ	**0.933**	**0.938**	**0.938**	**0.934**	**0.998**	**0.997**	**0.996**	**0.997**
β_3^2	0.903	0.907	0.907	0.907	0.996	0.994	0.994	0.995
RJB	0.926	0.931	0.931	0.930	0.996	0.995	0.994	0.996
X_{APD}	**0.916**	**0.922**	**0.922**	**0.921**	**0.996**	**0.995**	**0.994**	**0.996**
Z_{EPD}	0.903	0.909	0.909	0.909	0.996	0.994	0.994	0.996
$LF_{1,1}$	0.879	0.889	0.889	0.885	0.989	0.987	0.987	0.989

Source: author's work.

6. Symmetric distributions with constant non-zero excess kurtosis

Table 8 shows GoFT powers for group II of alternatives. The top three GoFTs by average power are marked in bold.

Table 8. GoFT powers for group II of alternatives

Specification	$n = 25$				$n = 50$			
				$\boldsymbol{AS(a), \bar{\gamma}_2 = -1.5}$				
a	1.317	1.954	2.708	6.333	1.317	1.954	2.708	6.333
$M(a;0,1)$	0.651	0.600	0.500	0.250	0.651	0.600	0.500	0.250
AD	0.762	0.760	0.755	0.760	0.991	0.990	0.992	0.991
SW	0.869	0.865	0.863	0.865	0.999	0.999	0.999	0.999
KT	0.713	0.709	0.714	0.707	0.980	0.979	0.978	0.976
AS	0.010	0.010	0.010	0.012	0.005	0.006	0.006	0.006
SF	0.667	0.669	0.665	0.667	0.994	0.990	0.992	0.993
AP	0.699	0.697	0.693	0.690	0.995	0.994	0.994	0.993
RJ	0.650	0.652	0.647	0.647	0.992	0.989	0.991	0.992
JB	0.005	0.005	0.005	0.006	0.374	0.380	0.378	0.370
H1	0.842	0.841	0.840	0.844	0.997	0.997	0.996	0.996
CS	**0.892**	**0.888**	**0.888**	**0.890**	**1.000**	**0.999**	**1.000**	**0.999**
AJB	0.002	0.003	0.001	0.002	0.103	0.104	0.100	0.106
BS	0.662	0.656	0.653	0.648	0.958	0.957	0.955	0.955
ZA	0.857	0.852	0.848	0.849	1.000	1.000	1.000	0.999
ZC	**0.909**	**0.904**	**0.905**	**0.905**	**1.000**	**1.000**	**1.000**	**1.000**

Table 8. GoFT powers for group II of alternatives (cont.)

Specification	$n = 25$				$n = 50$			
				$AS(a), \overline{\gamma}_2 = -1.5$ **(cont.)**				
SJ	0.002	0.002	0.001	0.001	0.000	0.000	0.000	0.000
β_3^2	**0.909**	**0.904**	**0.911**	**0.904**	**1.000**	**0.999**	**1.000**	**1.000**
RJB	0.003	0.003	0.003	0.003	0.000	0.001	0.000	0.000
X_{APD}	0.731	0.732	0.728	0.727	0.983	0.984	0.982	0.983
Z_{EPD}	0.779	0.780	0.780	0.775	0.990	0.989	0.987	0.987
$LF_{0,0}$	0.512	0.505	0.508	0.508	0.856	0.859	0.853	0.852
				$BL(a), \overline{\gamma}_2 = 1/3$				
a	0.437	0.678	1.163	2.357	0.437	0.678	1.163	2.357
$M(a; 0, 1)$	0.912	0.750	0.500	0.250	0.912	0.750	0.500	0.250
AD	0.095	0.096	0.093	0.095	0.132	0.129	0.134	0.134
SW	0.097	0.097	0.094	0.100	0.134	0.129	0.138	0.132
KT	0.094	0.090	0.091	0.088	0.115	0.121	0.125	0.114
AS	0.084	0.085	0.085	0.086	0.105	0.108	0.114	0.104
SF	**0.098**	**0.098**	**0.096**	**0.100**	**0.143**	**0.140**	**0.149**	**0.140**
AP	0.093	0.094	0.091	0.090	0.117	0.122	0.126	0.116
RJ	0.092	0.093	0.092	0.094	0.138	0.134	0.142	0.134
JB	0.087	0.087	0.086	0.086	0.115	0.116	0.123	0.111
H1	0.079	0.074	0.075	0.078	0.082	0.080	0.081	0.080
CS	**0.100**	**0.101**	**0.097**	**0.102**	**0.131**	**0.127**	**0.140**	**0.131**
AJB	0.086	0.086	0.085	0.086	0.116	0.116	0.125	0.112
BS	0.087	0.081	0.088	0.088	0.111	0.121	0.113	0.115
ZA	0.097	0.097	0.096	0.101	0.123	0.123	0.131	0.121
ZC	0.099	0.098	0.096	0.099	0.129	0.127	0.134	0.127
SJ	0.049	0.048	0.052	0.049	0.041	0.041	0.037	0.036
β_3^2	0.069	0.069	0.069	0.069	0.080	0.082	0.082	0.083
RJB	0.074	0.073	0.071	0.072	0.087	0.089	0.093	0.085
X_{APD}	0.093	0.089	0.089	0.091	0.110	0.112	0.114	0.109
Z_{EPD}	0.097	0.092	0.096	0.093	0.108	0.115	0.106	0.109
$LF_{1,1}$	**0.091**	**0.093**	**0.095**	**0.096**	**0.147**	**0.147**	**0.148**	**0.150**
				$BN(a), \overline{\gamma}_2 = -4/3$				
a	0.819	1.187	1.462	2.593	0.819	1.187	1.462	2.593
$M(a; 0, 1)$	0.699	0.600	0.500	0.250	0.699	0.600	0.500	0.250
AD	**0.832**	**0.832**	**0.834**	**0.826**	**0.998**	**0.997**	**0.997**	**0.998**
SW	0.721	0.721	0.723	0.710	0.984	0.983	0.983	0.985
KT	0.579	0.579	0.566	0.567	0.897	0.891	0.889	0.891
AS	0.015	0.015	0.015	0.016	0.008	0.010	0.010	0.010
SF	0.538	0.538	0.540	0.528	0.952	0.951	0.954	0.957
AP	0.552	0.552	0.539	0.539	0.941	0.937	0.937	0.936
RJ	0.516	0.516	0.518	0.506	0.946	0.947	0.949	0.952
JB	0.006	0.006	0.007	0.007	0.126	0.128	0.132	0.134
H1	0.639	0.639	0.633	0.618	0.974	0.975	0.975	0.976
CS	0.757	0.757	0.760	0.749	0.987	0.987	0.988	0.990
AJB	0.003	0.003	0.003	0.003	0.023	0.025	0.023	0.021
BS	**0.825**	**0.825**	**0.816**	**0.818**	**0.986**	**0.986**	**0.983**	**0.988**
ZA	0.502	0.502	0.504	0.492	0.903	0.903	0.907	0.906
ZC	0.650	0.650	0.647	0.639	0.945	0.948	0.948	0.947
SJ	0.007	0.007	0.008	0.009	0.001	0.001	0.001	0.001
β_3^2	0.524	0.524	0.519	0.514	0.884	0.885	0.886	0.881
RJB	0.007	0.007	0.007	0.008	0.002	0.002	0.001	0.002
X_{APD}	**0.827**	**0.827**	**0.828**	**0.826**	**0.994**	**0.994**	**0.993**	**0.994**
Z_{EPD}	0.809	0.809	0.803	0.804	0.984	0.985	0.983	0.985
$LF_{0,0}$	0.775	0.775	0.773	0.761	0.990	0.988	0.988	0.989

Table 8. GoFT powers for group II of alternatives (cont.)

Specification	$n = 25$				$n = 50$			
				$\boldsymbol{COS(a), \overline{\gamma}_2 = -0.806}$				
a	1.298	2.192	3.988	9.705	1.298	2.192	3.988	9.705
$M(a;0,1)$	0.937	0.750	0.500	0.250	0.937	0.750	0.500	0.250
AD	0.059	0.058	0.059	0.058	0.106	0.111	0.111	0.111
SW	0.054	0.052	0.053	0.052	0.114	0.123	0.119	0.124
KT	0.056	0.051	0.057	0.054	0.132	0.133	0.133	0.131
AS	0.005	0.006	0.007	0.007	0.003	0.003	0.003	0.004
SF	0.022	0.024	0.023	0.023	0.041	0.042	0.043	0.044
AP	0.041	0.036	0.041	0.037	0.164	0.166	0.170	0.166
RJ	0.019	0.021	0.020	0.021	0.037	0.038	0.040	0.041
JB	0.003	0.004	0.003	0.003	0.001	0.001	0.001	0.002
H1	0.053	0.050	0.055	0.054	0.135	0.137	0.139	0.138
CS	0.063	0.063	0.064	0.062	0.144	0.153	0.154	0.156
AJB	0.002	0.003	0.002	0.002	0.001	0.000	0.000	0.001
BS	**0.086**	**0.083**	**0.089**	**0.085**	**0.176**	**0.178**	**0.189**	**0.184**
ZA	0.037	0.036	0.037	0.037	0.106	0.109	0.110	0.114
ZC	0.054	0.053	0.055	0.053	0.130	0.132	0.132	0.138
SJ	0.005	0.005	0.006	0.006	0.001	0.001	0.002	0.001
β_3^2	**0.088**	**0.082**	**0.087**	**0.089**	**0.249**	**0.241**	**0.251**	**0.250**
RJB	0.003	0.003	0.004	0.003	0.001	0.001	0.000	0.001
X_{APD}	0.057	0.055	0.059	0.055	0.128	0.132	0.135	0.130
Z_{EPD}	**0.118**	**0.108**	**0.121**	**0.116**	**0.249**	**0.249**	**0.257**	**0.258**
$LF_{0,0}$	0.065	0.065	0.066	0.062	0.082	0.088	0.088	0.093
				$\boldsymbol{HS(a), \overline{\gamma}_2 = 2}$				
a	0.757	0.419	0.213	0.079	0.757	0.419	0.213	0.079
$M(a;0,1)$	0.944	0.750	0.500	0.250	0.944	0.750	0.500	0.250
AD	0.188	0.185	0.182	0.182	0.289	0.293	0.283	0.291
SW	0.201	0.200	0.197	0.196	0.318	0.325	0.317	0.323
KT	0.243	0.243	0.243	0.242	0.402	0.408	0.404	0.412
AS	0.209	0.211	0.210	0.206	0.266	0.268	0.272	0.271
SF	0.248	0.247	0.241	0.244	0.388	0.394	0.386	0.389
AP	0.227	0.236	0.229	0.228	0.336	0.343	0.340	0.342
RJ	0.239	0.240	0.233	0.235	0.379	0.384	0.376	0.380
JB	0.246	0.254	0.248	0.249	0.388	0.395	0.388	0.395
H1	0.228	0.227	0.222	0.225	0.347	0.351	0.348	0.357
CS	0.191	0.192	0.187	0.186	0.289	0.297	0.287	0.294
AJB	**0.256**	**0.262**	**0.256**	**0.255**	**0.400**	**0.408**	**0.401**	**0.408**
BS	0.203	0.198	0.193	0.197	0.356	0.354	0.355	0.357
ZA	0.205	0.207	0.197	0.196	0.283	0.287	0.281	0.289
ZC	0.202	0.204	0.197	0.196	0.301	0.305	0.303	0.309
SJ	**0.272**	**0.271**	**0.267**	**0.271**	**0.445**	**0.443**	**0.440**	**0.447**
β_3^2	0.229	0.222	0.225	0.226	0.382	0.388	0.382	0.388
RJB	**0.272**	**0.274**	**0.272**	**0.270**	**0.429**	**0.438**	**0.434**	**0.438**
X_{APD}	0.235	0.238	0.231	0.232	0.381	0.389	0.384	0.389
Z_{EPD}	0.210	0.210	0.205	0.205	0.370	0.377	0.370	0.373
$LF_{1,1}$	0.165	0.162	0.153	0.155	0.228	0.226	0.233	0.232
				$\boldsymbol{IUP(a), \overline{\gamma}_2 = -0.857}$				
a	1.966	3.304	5.999	14.567	1.966	3.304	5.999	14.567
$M(a;0,1)$	0.927	0.75	0.500	0.250	0.927	0.750	0.500	0.250
AD	0.068	0.072	0.065	0.067	0.136	0.130	0.132	0.132
SW	0.066	0.071	0.061	0.063	0.155	0.152	0.154	0.149
KT	0.070	0.071	0.068	0.067	0.170	0.171	0.170	0.171
AS	0.006	0.006	0.005	0.007	0.002	0.002	0.002	0.003

Table 8. GoFT powers for group II of alternatives (cont.)

Specification	$n=25$				$n=50$			
	$IUP(a), \bar{\gamma}_2=-0.857$ (cont.)							
SF	0.025	0.026	0.023	0.026	0.056	0.053	0.051	0.057
AP	0.050	0.051	0.049	0.048	0.211	0.210	0.213	0.214
RJ	0.022	0.023	0.021	0.024	0.051	0.049	0.046	0.052
JB	0.002	0.002	0.003	0.003	0.000	0.000	0.001	0.001
H1	0.063	0.067	0.062	0.064	0.175	0.168	0.173	0.169
CS	0.078	0.084	0.074	0.076	0.195	0.195	0.194	0.191
AJB	0.001	0.002	0.002	0.002	0.000	0.000	0.000	0.000
BS	**0.102**	**0.104**	**0.100**	**0.101**	**0.219**	**0.215**	**0.220**	**0.207**
ZA	0.043	0.045	0.042	0.042	0.144	0.144	0.141	0.140
ZC	0.064	0.071	0.063	0.064	0.178	0.176	0.172	0.177
SJ	0.003	0.004	0.004	0.005	0.001	0.001	0.001	0.001
β_3^2	**0.107**	**0.108**	**0.103**	**0.101**	**0.309**	**0.308**	**0.305**	**0.316**
RJB	0.002	0.002	0.002	0.003	0.000	0.000	0.001	0.000
X_{APD}	0.069	0.073	0.069	0.068	0.169	0.165	0.167	0.162
Z_{EPD}	**0.138**	**0.137**	**0.138**	**0.132**	**0.308**	**0.301**	**0.304**	**0.300**
$LF_{0,0}$	0.072	0.073	0.069	0.073	0.103	0.099	0.102	0.099
	$L(a), \bar{\gamma}_2=3$							
a	0.938	1.564	3.209	8.707	0.938	1.564	3.209	8.707
$M(a;0,1)$	0.888	0.75	0.5	0.25	0.888	0.750	0.500	0.250
AD	0.323	0.324	0.326	0.323	0.546	0.549	0.541	0.544
SW	0.309	0.312	0.316	0.313	0.517	0.518	0.517	0.515
KT	0.351	0.357	0.354	0.355	0.584	0.589	0.575	0.587
AS	0.283	0.284	0.290	0.284	0.353	0.350	0.349	0.347
SF	0.373	0.378	0.382	0.379	0.596	0.600	0.592	0.596
AP	0.321	0.327	0.328	0.328	0.483	0.484	0.485	0.485
RJ	0.365	0.368	0.371	0.370	0.587	0.590	0.582	0.587
JB	0.347	0.354	0.359	0.356	0.548	0.554	0.548	0.548
H1	**0.383**	**0.385**	**0.389**	**0.386**	**0.616**	**0.620**	**0.605**	**0.614**
CS	0.293	0.292	0.298	0.297	0.482	0.481	0.480	0.476
AJB	0.362	0.368	0.370	0.368	0.567	0.572	0.565	0.568
BS	0.355	0.357	0.349	0.355	0.637	0.647	0.624	0.634
ZA	0.296	0.300	0.305	0.303	0.449	0.449	0.449	0.446
ZC	0.287	0.287	0.293	0.290	0.454	0.454	0.454	0.449
SJ	**0.459**	**0.456**	**0.456**	**0.458**	**0.730**	**0.739**	**0.721**	**0.729**
β_3^2	0.363	0.360	0.359	0.366	0.605	0.615	0.590	0.607
RJB	**0.415**	**0.422**	**0.422**	**0.417**	**0.658**	**0.659**	**0.645**	**0.653**
X_{APD}	0.370	0.373	0.374	0.375	0.614	0.618	0.602	0.612
Z_{EPD}	0.339	0.340	0.338	0.343	0.609	0.616	0.593	0.604
$LF_{1,1}$	0.295	0.291	0.298	0.293	0.475	0.472	0.470	0.468
	$LOG(a), \bar{\gamma}_2=1.2$							
a	0.615	1.035	1.940	4.816	0.615	1.035	1.940	4.816
$M(a;0,1)$	0.967	0.750	0.500	0.250	0.967	0.750	0.500	0.250
AD	0.116	0.119	0.113	0.112	0.154	0.159	0.151	0.156
SW	0.130	0.132	0.127	0.126	0.189	0.196	0.189	0.190
KT	0.162	0.161	0.154	0.158	0.260	0.259	0.259	0.256
AS	0.149	0.147	0.148	0.144	0.182	0.189	0.185	0.182
SF	0.161	0.165	0.159	0.161	0.241	0.242	0.236	0.238
AP	0.160	0.156	0.155	0.154	0.218	0.218	0.219	0.218
RJ	0.154	0.159	0.153	0.154	0.233	0.235	0.229	0.231
JB	0.169	0.170	0.165	0.166	0.253	0.253	0.252	0.252
H1	0.139	0.144	0.141	0.144	0.191	0.202	0.193	0.188
CS	0.125	0.125	0.121	0.122	0.170	0.180	0.171	0.171

Table 8. GoFT powers for group II of alternatives (cont.)

Specification	$n = 25$				$n = 50$			
				$\boldsymbol{LOG(a), \bar{\gamma}_2 = 1.2}$ **(cont.)**				
AJB	**0.174**	**0.174**	**0.171**	**0.170**	**0.264**	**0.264**	**0.259**	**0.260**
BS	0.120	0.118	0.115	0.121	0.191	0.195	0.195	0.192
ZA	0.135	0.137	0.133	0.129	0.172	0.181	0.173	0.171
ZC	0.138	0.136	0.133	0.130	0.190	0.197	0.194	0.194
SJ	**0.163**	**0.168**	**0.169**	**0.170**	**0.262**	**0.259**	**0.261**	**0.257**
β_3^2	0.142	0.141	0.134	0.142	0.228	0.228	0.223	0.225
RJB	**0.176**	**0.177**	**0.174**	**0.179**	**0.274**	**0.273**	**0.269**	**0.268**
X_{APD}	0.149	0.152	0.146	0.148	0.230	0.232	0.225	0.227
Z_{EPD}	0.130	0.131	0.127	0.132	0.213	0.209	0.210	0.206
$LF_{1,1}$	0.100	0.105	0.100	0.098	0.125	0.130	0.123	0.124
				$\boldsymbol{RCOS(a), \bar{\gamma}_2 = -0.594}$				
a	2.551	1.578	0.883	0.381	2.551	1.578	0.883	0.381
$M(a; 0, 1)$	0.965	0.750	0.500	0.250	0.965	0.750	0.500	0.250
AD	0.046	0.044	0.047	0.047	0.060	0.061	0.062	0.059
SW	0.037	0.041	0.040	0.041	0.056	0.056	0.053	0.053
KT	0.028	0.031	0.031	0.030	0.046	0.044	0.046	0.049
AS	0.011	0.011	0.012	0.011	0.005	0.007	0.005	0.007
SF	0.022	0.020	0.022	0.022	0.021	0.021	0.022	0.021
AP	0.023	0.024	0.025	0.021	0.061	0.059	0.059	0.064
RJ	0.019	0.017	0.019	0.020	0.019	0.020	0.020	0.019
JB	0.007	0.007	0.008	0.007	0.001	0.002	0.003	0.002
H1	0.035	0.039	0.036	0.035	0.062	0.060	0.061	0.064
CS	0.042	0.046	0.045	0.047	0.068	0.071	0.067	0.068
AJB	0.006	0.005	0.007	0.006	0.001	0.001	0.002	0.001
BS	**0.055**	**0.054**	**0.056**	**0.050**	**0.094**	**0.091**	**0.090**	**0.094**
ZA	0.029	0.030	0.030	0.032	0.048	0.052	0.047	0.046
ZC	0.036	0.038	0.038	0.037	0.054	0.055	0.052	0.051
SJ	0.012	0.011	0.011	0.012	0.003	0.005	0.005	0.005
β_3^2	**0.045**	**0.045**	**0.047**	**0.046**	**0.096**	**0.095**	**0.091**	**0.097**
RJB	0.008	0.007	0.009	0.007	0.001	0.002	0.002	0.002
X_{APD}	0.038	0.036	0.038	0.036	0.060	0.061	0.061	0.062
Z_{EPD}	**0.066**	**0.066**	**0.068**	**0.065**	**0.118**	**0.115**	**0.114**	**0.121**
$LF_{0,0}$	0.056	0.054	0.050	0.053	0.060	0.061	0.061	0.062
				$\boldsymbol{SC(a), \bar{\gamma}_2 = -1}$				
a	1.716	1.183	0.689	0.321	1.716	1.183	0.689	0.321
$M(a; 0, 1)$	0.893	0.750	0.500	0.250	0.893	0.750	0.500	0.250
AD	0.102	0.103	0.096	0.103	0.244	0.251	0.240	0.239
SW	0.110	0.112	0.103	0.110	0.322	0.328	0.317	0.315
KT	0.129	0.126	0.127	0.122	0.361	0.351	0.355	0.353
AS	0.005	0.005	0.004	0.005	0.002	0.002	0.002	0.002
SF	0.039	0.041	0.038	0.042	0.135	0.131	0.127	0.124
AP	0.094	0.094	0.094	0.095	0.428	0.418	0.419	0.419
RJ	0.035	0.036	0.035	0.037	0.123	0.120	0.116	0.114
JB	0.002	0.002	0.002	0.002	0.001	0.001	0.001	0.001
H1	0.113	0.109	0.111	0.115	0.342	0.339	0.327	0.332
CS	**0.134**	**0.135**	**0.125**	**0.132**	**0.388**	**0.389**	**0.380**	**0.380**
AJB	0.002	0.001	0.000	0.001	0.000	0.000	0.000	0.000
BS	0.157	0.157	0.151	0.154	0.361	0.363	0.347	0.352
ZA	0.074	0.073	0.071	0.073	0.326	0.328	0.314	0.318
ZC	0.124	0.123	0.116	0.124	0.382	0.384	0.378	0.374
SJ	0.002	0.002	0.002	0.001	0.000	0.000	0.000	0.000
β_3^2	**0.192**	**0.193**	**0.193**	**0.189**	**0.570**	**0.566**	**0.561**	**0.565**

Table 8. GoFT powers for group II of alternatives (cont.)

Specification	$n = 25$				$n = 50$			
				$SC(a), \overline{\gamma}_2 = -1$ (cont.)				
RJB	0.002	0.002	0.001	0.002	0.000	0.000	0.000	0.000
X_{APD}	0.114	0.116	0.112	0.116	0.314	0.316	0.300	0.307
Z_{EPD}	**0.215**	**0.212**	**0.212**	**0.213**	**0.496**	**0.489**	**0.492**	**0.489**
$LF_{0,0}$	0.092	0.098	0.091	0.091	0.153	0.157	0.150	0.146
				$SIN(a), \overline{\gamma}_2 = -0.806$				
a	4.089	2.622	1.499	0.672	4.089	2.622	1.499	0.672
$M(a;0,1)$	0.937	0.750	0.500	0.250	0.937	0.750	0.500	0.250
AD	0.061	0.064	0.060	0.058	0.102	0.114	0.104	0.106
SW	0.057	0.058	0.056	0.054	0.113	0.120	0.112	0.119
KT	0.056	0.056	0.057	0.057	0.123	0.130	0.128	0.134
AS	0.007	0.007	0.005	0.006	0.003	0.003	0.002	0.004
SF	0.024	0.022	0.021	0.023	0.037	0.042	0.041	0.042
AP	0.040	0.040	0.040	0.042	0.158	0.169	0.161	0.169
RJ	0.021	0.020	0.020	0.020	0.033	0.039	0.036	0.039
JB	0.003	0.003	0.003	0.003	0.001	0.001	0.000	0.001
H1	0.056	0.053	0.054	0.054	0.129	0.136	0.132	0.138
CS	0.068	0.070	0.066	0.065	0.145	0.153	0.141	0.152
AJB	0.002	0.003	0.002	0.002	0.000	0.001	0.000	0.000
BS	**0.085**	**0.086**	**0.088**	**0.086**	**0.173**	**0.182**	**0.182**	**0.183**
ZA	0.036	0.038	0.037	0.035	0.101	0.107	0.105	0.109
ZC	0.057	0.057	0.057	0.053	0.127	0.134	0.125	0.133
SJ	0.005	0.005	0.006	0.005	0.001	0.001	0.001	0.001
β_3^2	**0.086**	**0.088**	**0.086**	**0.087**	**0.238**	**0.251**	**0.237**	**0.253**
RJB	0.004	0.003	0.004	0.003	0.000	0.001	0.000	0.000
X_{APD}	0.061	0.060	0.061	0.059	0.124	0.134	0.127	0.134
Z_{EPD}	**0.114**	**0.114**	**0.115**	**0.113**	**0.243**	**0.254**	**0.248**	**0.255**
$LF_{0,0}$	0.069	0.070	0.067	0.068	0.085	0.091	0.087	0.089
				$TRI(a), \overline{\gamma}_2 = -0.6$				
a	2.262	1.374	0.772	0.341	2.262	1.374	0.772	0.341
$M(a;0,1)$	0.962	0.750	0.500	0.250	0.962	0.750	0.500	0.250
AD	0.040	0.042	0.037	0.038	0.049	0.050	0.045	0.050
SW	0.033	0.034	0.034	0.033	0.047	0.047	0.045	0.046
KT	0.024	0.027	0.025	0.027	0.040	0.036	0.036	0.039
AS	0.008	0.009	0.009	0.009	0.004	0.004	0.004	0.004
SF	0.018	0.019	0.017	0.018	0.016	0.015	0.014	0.016
AP	0.019	0.018	0.018	0.020	0.054	0.048	0.049	0.052
RJ	0.016	0.017	0.016	0.016	0.015	0.014	0.013	0.014
JB	0.005	0.004	0.005	0.006	0.001	0.001	0.001	0.001
H1	0.032	0.031	0.029	0.030	0.048	0.043	0.042	0.043
CS	0.038	0.040	0.040	0.038	0.059	0.057	0.056	0.060
AJB	0.005	0.004	0.005	0.006	0.001	0.001	0.001	0.001
BS	**0.044**	**0.045**	**0.041**	**0.044**	**0.065**	**0.061**	**0.059**	**0.062**
ZA	0.024	0.025	0.024	0.024	0.042	0.041	0.040	0.040
ZC	0.032	0.032	0.032	0.031	0.050	0.048	0.046	0.048
SJ	0.014	0.016	0.013	0.013	0.007	0.008	0.006	0.009
β_3^2	**0.044**	**0.043**	**0.043**	**0.043**	**0.091**	**0.089**	**0.086**	**0.093**
RJB	0.007	0.007	0.007	0.007	0.001	0.002	0.001	0.001
X_{APD}	0.031	0.031	0.025	0.028	0.046	0.040	0.039	0.040
Z_{EPD}	**0.056**	**0.055**	**0.051**	**0.054**	**0.091**	**0.088**	**0.083**	**0.089**
$LF_{0,0}$	0.045	0.045	0.045	0.0454	0.048	0.052	0.045	0.051

Table 8. GoFT powers for group II of alternatives (cont.)

Specification	$n = 25$				$n = 50$			
				$\boldsymbol{U(a), \bar{\gamma}_2 = -1.2}$				
a	1.488	1.150	0.674	0.319	1.488	1.150	0.674	0.319
$M(a;0,1)$	0.815	0.750	0.50	0.250	0.815	0.750	0.500	0.250
AD	0.231	0.226	0.238	0.229	0.573	0.572	0.580	0.575
SW	0.288	0.282	0.292	0.289	0.748	0.747	0.755	0.750
KT	0.291	0.293	0.292	0.292	0.709	0.703	0.713	0.710
AS	0.005	0.004	0.004	0.006	0.003	0.003	0.002	0.002
SF	0.119	0.119	0.124	0.121	0.475	0.472	0.478	0.484
AP	0.245	0.244	0.244	0.242	0.781	0.773	0.779	0.777
RJ	0.110	0.109	0.115	0.111	0.456	0.453	0.458	0.463
JB	0.002	0.001	0.001	0.002	0.008	0.007	0.005	0.007
H1	0.292	0.286	0.294	0.291	0.709	0.703	0.709	0.704
CS	0.327	0.325	0.334	0.326	0.805	0.804	0.812	0.805
AJB	0.001	0.001	0.000	0.001	0.001	0.001	0.001	0.001
BS	0.281	0.284	0.289	0.285	0.621	0.623	0.624	0.619
ZA	0.226	0.223	0.231	0.224	0.791	0.790	0.797	0.792
ZC	**0.327**	**0.327**	**0.334**	**0.329**	**0.825**	**0.825**	**0.832**	**0.826**
SJ	0.001	0.001	0.001	0.001	0.000	0.000	0.000	0.000
β_3^2	**0.444**	**0.448**	**0.447**	**0.440**	**0.907**	**0.902**	**0.907**	**0.907**
RJB	0.001	0.001	0.001	0.002	0.000	0.000	0.000	0.000
X_{APD}	0.257	0.254	0.261	0.254	0.635	0.629	0.638	0.632
Z_{EPD}	**0.386**	**0.394**	**0.394**	**0.391**	**0.787**	**0.780**	**0.784**	**0.783**
$LF_{0,0}$	0.160	0.164	0.169	0.161	0.321	0.318	0.316	0.317
				$\boldsymbol{UP(a), \bar{\gamma}_2 = -1.81}$				
a	4/3	0.671	0.417	0.227	4/3	0.671	0.417	0.227
$M(a;0,1)$	0.446	0.350	0.250	0.150	0.446	0.350	0.250	0.150
AD	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
SW	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
KT	0.925	0.923	0.929	0.926	0.997	0.996	0.997	0.997
AS	0.031	0.032	0.031	0.032	0.018	0.019	0.017	0.015
SF	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
AP	0.955	0.954	0.958	0.956	1.000	1.000	1.000	1.000
RJ	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
JB	0.032	0.031	0.032	0.032	1.000	1.000	1.000	1.000
H1	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
CS	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
AJB	0.006	0.007	0.006	0.008	0.989	0.991	0.992	0.990
BS	0.958	0.956	0.961	0.958	0.999	0.999	0.999	1.000
ZA	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
ZC	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
SJ	0.017	0.017	0.019	0.018	0.002	0.002	0.001	0.001
β_3^2	0.975	0.972	0.974	0.973	1.000	1.000	1.000	1.000
RJB	0.015	0.015	0.016	0.016	0.005	0.006	0.004	0.004
X_{APD}	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**
Z_{EPD}	0.957	0.954	0.958	0.956	0.999	0.998	0.999	0.999
$LF_{0,0}$	0.997	0.997	0.998	0.997	1.000	1.000	1.000	1.000
				$\boldsymbol{V(a), \bar{\gamma}_2 = -5.3}$				
a	1.407	0.875	0.512	0.343	1.407	0.875	0.512	0.343
$M(a;0,1)$	0.570	0.50	0.350	0.250	0.570	0.500	0.350	0.250
AD	0.987	0.987	0.988	0.988	1.000	1.000	1.000	1.000
SW	0.992	0.992	0.993	0.993	1.000	1.000	1.000	1.000
KT	0.879	0.876	0.879	0.883	0.993	0.993	0.997	0.994

Table 8. GoFT powers for group II of alternatives (cont.)

Specification	$n = 25$				$n = 50$			
				$V(a), \overline{\gamma}_2 = -5.3$ (cont.)				
AS	0.019	0.019	0.022	0.017	0.013	0.011	0.015	0.011
SF	0.959	0.953	0.957	0.955	1.000	1.000	1.000	1.000
AP	0.904	0.902	0.904	0.909	1.000	1.000	1.000	0.000
RJ	0.952	0.948	0.950	0.947	1.000	1.000	1.000	1.000
JB	0.012	0.012	0.014	0.012	0.952	0.958	1.000	0.953
H1	0.991	0.991	0.992	0.992	1.000	1.000	1.000	1.000
CS	**0.994**	**0.995**	**0.995**	**0.995**	**1.000**	**1.000**	**1.000**	**1.000**
AJB	0.004	0.004	0.003	0.003	0.607	0.613	0.990	0.603
BS	0.921	0.919	0.918	0.924	0.997	0.997	1.000	0.998
ZA	0.985	0.984	0.985	0.984	1.000	1.000	1.000	1.000
ZC	**0.994**	**0.995**	**0.995**	**0.994**	**1.000**	**1.000**	**1.000**	**1.000**
SJ	0.006	0.007	0.007	0.006	0.001	0.001	0.001	0.000
β_3^2	0.955	0.957	0.951	0.957	0.999	1.000	1.000	1.000
RJB	0.008	0.008	0.008	0.008	0.002	0.002	0.004	0.002
X_{APD}	**0.993**	**0.995**	**0.994**	**0.994**	**1.000**	**1.000**	**1.000**	**1.000**
Z_{EPD}	0.933	0.932	0.930	0.936	0.998	0.998	0.999	0.998
$LF_{0,0}$	0.893	0.899	0.897	0.890	0.999	0.998	1.000	0.999

Source: author's work.

7. Asymmetric distributions with constant non-zero excess kurtosis

Table 9 shows GoFT powers for group III of alternatives. The top three GoFTs by average power are marked in bold.

Table 9. GoFT powers for group III of alternatives

Specification	$n = 25$				$n = 50$			
				$EXP(a), \gamma_1 = 2, \overline{\gamma}_2 = 6$				
a	2.000	3.577	5.645	10.039	2.000	3.577	5.645	10.039
$M(a;0,1)$	0.500	0.400	0.300	0.200	0.500	0.400	0.300	0.200
AD	0.872	0.871	0.881	0.873	0.997	0.997	0.997	0.997
SW	0.923	0.922	0.928	0.924	1.000	0.999	1.000	0.999
KT	0.473	0.478	0.473	0.477	0.713	0.713	0.715	0.720
AS	0.810	0.811	0.820	0.812	0.988	0.988	0.989	0.988
SF	0.892	0.893	0.904	0.895	0.999	0.998	0.999	0.998
AP	0.684	0.685	0.685	0.685	0.951	0.953	0.954	0.957
RJ	0.886	0.886	0.899	0.889	0.999	0.998	0.999	0.998
T1n	0.917	0.918	0.921	0.916	0.999	0.998	0.999	0.998
JB	0.732	0.739	0.740	0.737	0.976	0.975	0.976	0.978
H1	0.881	0.885	0.892	0.883	0.998	0.997	0.998	0.998
CC	0.851	0.848	0.857	0.849	0.995	0.995	0.995	0.996
CS	**0.926**	**0.925**	**0.931**	**0.927**	**1.000**	**0.999**	**1.000**	**1.000**
AJB	0.672	0.674	0.674	0.673	0.955	0.956	0.958	0.959
ZA	**0.944**	**0.946**	**0.950**	**0.945**	**1.000**	**1.000**	**1.000**	**1.000**
ZC	0.923	0.921	0.927	0.922	1.000	0.999	1.000	0.999
β_3^2	0.175	0.174	0.175	0.171	0.239	0.241	0.244	0.245
H_n	0.914	0.917	0.923	0.919	0.998	0.997	0.997	0.998
X_{APD}	0.841	0.840	0.849	0.840	0.995	0.994	0.995	0.995
B_v	**0.948**	**0.947**	**0.951**	**0.948**	**1.000**	**1.000**	**1.000**	**1.000**

Table 9. GoFT powers for group III of alternatives (cont.)

Specification	$n = 25$				$n = 50$			
			$GU(a), \gamma_1 = 1.14, \overline{\gamma}_2 = 2.4$					
$LF_{0,1}$	0.768	0.771	0.774	0.769	0.968	0.968	0.969	0.970
δ	0.902	0.902	0.910	0.904	0.997	0.996	0.997	0.997
a	0.920	0.534	0.278	0.108	0.920	0.534	0.278	0.108
$M(a;0,1)$	0.868	0.750	0.500	0.250	0.868	0.750	0.500	0.250
AD	0.335	0.336	0.339	0.339	0.597	0.595	0.600	0.603
SW	0.384	0.388	0.397	0.393	0.688	0.685	0.692	0.687
KT	0.235	0.241	0.242	0.245	0.390	0.384	0.392	0.391
AS	0.396	0.399	0.409	0.409	0.699	0.701	0.704	0.703
SF	0.379	0.386	0.394	0.391	0.671	0.667	0.674	0.670
AP	0.329	0.337	0.342	0.338	0.583	0.581	0.584	0.583
RJ	0.368	0.376	0.384	0.381	0.660	0.658	0.665	0.661
T1n	**0.427**	**0.433**	**0.438**	**0.437**	**0.746**	**0.742**	**0.745**	**0.743**
JB	0.348	0.354	0.360	0.359	0.615	0.610	0.614	0.617
H1	0.344	0.348	0.354	0.353	0.630	0.633	0.631	0.637
CC	**0.405**	**0.409**	**0.419**	**0.419**	**0.718**	**0.722**	**0.721**	**0.720**
CS	0.381	0.386	0.392	0.390	0.686	0.684	0.692	0.684
AJB	0.324	0.332	0.333	0.332	0.583	0.580	0.584	0.583
ZA	**0.396**	**0.398**	**0.408**	**0.404**	**0.703**	**0.703**	**0.707**	**0.699**
ZC	0.381	0.388	0.394	0.390	0.672	0.676	0.679	0.673
β_3^2	0.108	0.110	0.110	0.118	0.158	0.161	0.163	0.160
H_n	0.393	0.401	0.403	0.403	0.625	0.625	0.629	0.626
X_{APD}	0.335	0.340	0.348	0.346	0.637	0.639	0.643	0.644
B_v	0.305	0.307	0.310	0.306	0.540	0.533	0.541	0.537
$LF_{0,1}$	0.322	0.329	0.328	0.329	0.523	0.518	0.529	0.528
δ	0.391	0.396	0.401	0.400	0.636	0.641	0.643	0.641
			$HL(a), \gamma_1 = 1.54, \overline{\gamma}_2 = 3.584$					
a	1.000	0.288	0.185	0.106	1.000	0.288	0.185	0.106
$M(a;0,1)$	0.500	0.400	0.300	0.200	0.500	0.400	0.300	0.200
AD	0.674	0.669	0.671	0.671	0.954	0.950	0.949	0.952
SW	0.765	0.762	0.760	0.759	0.985	0.986	0.984	0.985
KT	0.335	0.346	0.341	0.337	0.528	0.521	0.532	0.527
AS	0.640	0.637	0.637	0.635	0.933	0.927	0.928	0.931
SF	0.712	0.709	0.710	0.713	0.974	0.974	0.971	0.974
AP	0.508	0.511	0.512	0.511	0.830	0.822	0.831	0.826
RJ	0.702	0.698	0.697	0.702	0.972	0.972	0.970	0.972
T1n	0.766	0.770	0.765	0.775	0.976	0.979	0.977	0.980
JB	0.547	0.548	0.549	0.548	0.879	0.869	0.875	0.877
H1	0.690	0.682	0.685	0.685	0.964	0.965	0.962	0.965
CC	0.673	0.670	0.673	0.669	0.956	0.955	0.953	0.955
CS	**0.770**	**0.766**	**0.764**	**0.767**	**0.987**	**0.988**	**0.986**	**0.987**
AJB	0.491	0.495	0.493	0.491	0.833	0.823	0.834	0.832
ZA	**0.808**	**0.807**	**0.807**	**0.807**	**0.994**	**0.995**	**0.995**	**0.995**
ZC	0.763	0.761	0.756	0.759	0.986	0.987	0.985	0.986
β_3^2	0.122	0.123	0.120	0.117	0.150	0.155	0.152	0.152
H_{nv}	0.751	0.746	0.755	0.758	0.959	0.961	0.958	0.962
X_{APD}	0.636	0.638	0.637	0.644	0.947	0.946	0.945	0.947
B_v	**0.807**	**0.802**	**0.804**	**0.803**	**0.989**	**0.990**	**0.992**	**0.990**
$LF_{0,1}$	0.565	0.568	0.568	0.574	0.840	0.832	0.840	0.836
δ	0.725	0.725	0.730	0.730	0.954	0.952	0.951	0.953
			$HN(a), \gamma_1 = 0.995, \overline{\gamma}_2 = 0.869$					
a	2.000	3.253	5.029	8.668	2.000	3.253	5.029	8.668
$M(a;0,1)$	0.500	0.400	0.300	0.200	0.500	0.400	0.300	0.200

Table 9. GoFT powers for group III of alternatives (cont.)

Specification	$n = 25$				$n = 50$			
	$HN(a), \gamma_1 = 0.995, \overline{\gamma}_2 = 0.869$ (cont.)							
AD	0.468	0.461	0.472	0.470	0.823	0.836	0.832	0.836
SW	0.568	0.571	0.570	0.568	0.930	0.931	0.932	0.932
KT	0.178	0.181	0.174	0.175	0.248	0.246	0.243	0.237
AS	0.414	0.419	0.413	0.411	0.755	0.761	0.765	0.757
SF	0.497	0.501	0.499	0.495	0.884	0.886	0.885	0.887
AP	0.304	0.305	0.305	0.299	0.572	0.577	0.581	0.569
RJ	0.483	0.488	0.485	0.481	0.875	0.879	0.878	0.879
T_{1n}	0.568	0.576	0.575	0.570	0.899	0.899	0.903	0.904
JB	0.330	0.327	0.328	0.326	0.634	0.637	0.640	0.633
H_1	0.481	0.479	0.481	0.480	0.861	0.866	0.866	0.869
CC	0.449	0.453	0.452	0.448	0.819	0.824	0.825	0.821
CS	**0.577**	**0.581**	**0.577**	**0.579**	**0.939**	**0.940**	**0.940**	**0.942**
AJB	0.280	0.277	0.281	0.278	0.553	0.560	0.564	0.557
ZA	**0.631**	**0.633**	**0.631**	**0.625**	**0.973**	**0.972**	**0.975**	**0.973**
ZC	0.567	0.572	0.568	0.568	0.936	0.936	0.938	0.937
β_3^2	0.080	0.084	0.076	0.075	0.110	0.110	0.104	0.105
H_n	0.577	0.584	0.578	0.578	0.858	0.865	0.867	0.873
X_{APD}	0.429	0.431	0.432	0.430	0.811	0.818	0.817	0.818
B_v	**0.650**	**0.658**	**0.649**	**0.646**	**0.957**	**0.954**	**0.959**	**0.960**
$LF_{0,1}$	0.398	0.401	0.403	0.402	0.656	0.661	0.659	0.655
δ	0.531	0.527	0.531	0.530	0.828	0.835	0.835	0.839
	$LW(a), \gamma_1 = -1.14, \overline{\gamma}_2 = 2.4$							
a	0.920	0.534	0.278	0.108	0.920	0.534	0.278	0.108
$M(a; 0, 1)$	0.868	0.750	0.500	0.250	0.868	0.750	0.500	0.250
AD	0.339	0.333	0.335	0.338	0.605	0.596	0.604	0.594
SW	0.399	0.387	0.394	0.392	0.694	0.685	0.693	0.682
KT	0.246	0.238	0.241	0.236	0.393	0.380	0.393	0.390
AS	**0.412**	**0.403**	**0.401**	**0.399**	**0.708**	**0.700**	**0.707**	**0.697**
SF	0.394	0.387	0.389	0.385	0.676	0.667	0.677	0.669
AP	0.344	0.336	0.337	0.332	0.592	0.578	0.591	0.583
RJ	0.384	0.377	0.379	0.374	0.668	0.659	0.668	0.659
T1n	**0.439**	**0.440**	**0.439**	**0.436**	**0.749**	**0.745**	**0.749**	**0.744**
JB	0.365	0.354	0.355	0.352	0.620	0.608	0.620	0.610
H1	0.353	0.349	0.355	0.348	0.637	0.627	0.640	0.628
CC	0.411	0.400	0.398	0.397	0.707	0.697	0.707	0.696
CS	0.395	0.384	0.388	0.388	0.691	0.681	0.691	0.682
AJB	0.339	0.328	0.329	0.325	0.591	0.578	0.591	0.581
ZA	**0.409**	**0.400**	**0.405**	**0.401**	**0.711**	**0.702**	**0.707**	**0.704**
ZC	0.393	0.385	0.388	0.386	0.685	0.672	0.682	0.673
β_3^2	0.115	0.109	0.116	0.118	0.161	0.153	0.160	0.153
H_n	0.205	0.205	0.205	0.206	0.443	0.433	0.444	0.434
X_{APD}	0.345	0.340	0.344	0.338	0.649	0.636	0.646	0.637
B_v	0.309	0.309	0.306	0.303	0.543	0.531	0.547	0.539
$LF_{1,0}$	0.328	0.329	0.322	0.328	0.528	0.524	0.530	0.527
δ	0.209	0.213	0.214	0.214	0.462	0.457	0.464	0.456
	$MX(a), \gamma_1 = 0.486, \overline{\gamma}_2 = 0.108$							
a	0.665	0.508	0.368	0.230	0.665	0.508	0.368	0.230
$M(a; 1, 0.458)$	0.950	0.750	0.500	0.250	0.950	0.750	0.500	0.250
AD	0.104	0.105	0.107	0.101	0.173	0.177	0.178	0.173
SW	0.121	0.121	0.126	0.117	0.224	0.230	0.235	0.227
KT	0.073	0.077	0.079	0.080	0.093	0.091	0.095	0.094
AS	0.121	0.118	0.126	0.117	0.231	0.232	0.238	0.221

Table 9. GoFT powers for group III of alternatives (cont.)

Specification	$n = 25$				$n = 50$			
$\boldsymbol{MX(a), \gamma_1 = 0.486, \bar{\gamma}_2 = 0.108}$ **(cont.)**								
SF	0.113	0.113	0.119	0.111	0.202	0.204	0.207	0.196
AP	0.100	0.102	0.105	0.102	0.171	0.167	0.166	0.163
RJ	0.107	0.107	0.112	0.105	0.194	0.195	0.197	0.188
T1n	**0.134**	**0.133**	**0.143**	**0.129**	**0.272**	**0.274**	**0.275**	**0.263**
JB	0.102	0.100	0.105	0.100	0.169	0.167	0.173	0.161
H1	0.100	0.102	0.104	0.102	0.188	0.187	0.191	0.182
CC	0.125	0.121	0.128	0.120	0.248	0.247	0.253	0.237
CS	0.121	0.122	0.126	0.118	0.227	0.236	0.238	0.231
AJB	0.091	0.093	0.096	0.092	0.152	0.151	0.156	0.145
ZA	**0.122**	**0.124**	**0.131**	**0.119**	**0.250**	**0.255**	**0.257**	**0.249**
ZC	0.119	0.122	0.127	0.117	0.224	0.224	0.228	0.222
β_3^2	0.046	0.048	0.052	0.050	0.053	0.051	0.052	0.055
H_n	**0.150**	**0.153**	**0.155**	**0.153**	**0.219**	**0.223**	**0.226**	**0.222**
X_{APD}	0.098	0.099	0.104	0.100	0.189	0.191	0.191	0.187
B_v	0.108	0.110	0.113	0.107	0.193	0.201	0.204	0.198
$LF_{0,1}$	0.125	0.135	0.131	0.131	0.185	0.189	0.187	0.183
δ	0.136	0.133	0.137	0.133	0.207	0.207	0.207	0.203
$\boldsymbol{RY(a), \gamma_1 = 0.631, \bar{\gamma}_2 = 0.245}$								
a	0.885	1.187	1.737	2.961	0.885	1.187	1.737	2.961
$M(a; 1, 0.595)$	0.925	0.750	0.500	0.250	0.925	0.750	0.500	0.250
AD	0.156	0.156	0.156	0.163	0.300	0.316	0.303	0.308
SW	0.188	0.192	0.194	0.199	0.411	0.414	0.410	0.413
KT	0.094	0.100	0.099	0.100	0.125	0.124	0.126	0.121
AS	0.178	0.175	0.178	0.184	0.368	0.363	0.365	0.358
SF	0.170	0.172	0.173	0.178	0.357	0.361	0.353	0.355
AP	0.136	0.139	0.142	0.146	0.255	0.254	0.254	0.251
RJ	0.160	0.165	0.166	0.169	0.345	0.349	0.341	0.343
T1n	**0.216**	**0.214**	**0.217**	**0.221**	**0.449**	**0.454**	**0.451**	**0.453**
JB	0.137	0.140	0.144	0.147	0.261	0.261	0.263	0.259
H1	0.150	0.153	0.156	0.160	0.336	0.342	0.334	0.335
CC	0.184	0.184	0.186	0.192	0.396	0.392	0.391	0.391
CS	**0.192**	**0.194**	**0.196**	**0.199**	**0.418**	**0.427**	**0.420**	**0.421**
AJB	0.121	0.124	0.125	0.132	0.229	0.230	0.235	0.228
ZA	**0.202**	**0.202**	**0.205**	**0.211**	**0.474**	**0.480**	**0.472**	**0.474**
ZC	0.187	0.188	0.192	0.197	0.407	0.416	0.408	0.410
β_3^2	0.052	0.055	0.053	0.059	0.064	0.070	0.063	0.063
H_n	0.222	0.223	0.221	0.229	0.365	0.368	0.361	0.371
X_{APD}	0.148	0.147	0.153	0.156	0.327	0.334	0.327	0.332
B_v	0.178	0.179	0.183	0.184	0.395	0.410	0.403	0.398
$LF_{0,1}$	0.173	0.172	0.177	0.182	0.270	0.279	0.275	0.280
δ	0.199	0.198	0.198	0.206	0.341	0.345	0.342	0.345

Source: author's work.

Regarding the Monte Carlo simulation results, the powers for a given test and a given sample size (written in rows) are very similar or even identical when the SM is changing. Let p_{min} and p_{max} be the smallest and the largest powers of the analysed GoFT, respectively. The commonly known Z test was used to examine the equality of p_{min} and p_{max}. For all the analysed groups of alternatives and GoFTs, hypothesis $H_0 \colon p_{min} = p_{max}$ was adopted.

The SJ, RJB ($\bar{\gamma}_2 > 0$), β_3^2 and CS tests ($\bar{\gamma}_2 < 0$) are most efficient for group I of alternatives, the SJ, H1 and RJB tests for group II, and the ZA and CS tests for group III. Table 10 provides more details. The AS, JB, AJB, SJ, RJB tests are not recommended for alternatives with $\bar{\gamma}_2 < 0$.

Table 10. Summary of the results from Tables 7–9 for groups I–III of the alternatives (A)

Alternatives	Best tests
Group I	
C	SJ, H1, RJB
SL	SJ, RJB, H1
Group II	
AS^-	ZC, β_3^2, CS
BL^+	SF, $LF_{1,1}$, CS
BN^-	AD, X_{APD}, BS
COS^-	Z_{EPD}, β_3^2, BS
HS^+	SJ, RJB, AJB
IUP^-	Z_{EPD}, β_3^2, BS
L^+	SJ, RJB, H1
LOG^+	RJB, AJB, SJ
$RCOS^-$	Z_{EPD}, BS, β_3^2
SC^-	β_3^2, Z_{EPD}, CS
SIN^-	Z_{EPD}, β_3^2, BS
TRI^-	Z_{EPD}, β_3^2, BS
U^-	β_3^2, Z_{EPD}, ZC
UP^-	AD, SW, SF, RJ, H1, CS, ZA, ZC, X_{APD}
Group III	
V^-	X_{APD}, ZC, CS
EXP	B_v, ZA, CS
GU	$T1_n$, CC, ZA
HL	ZA, B_v, CS
HN	B_v, ZA, CS
LW	$T1_n$, ZA, AS
MX	$T1_n$, H_n, ZA
RY	ZA, $T1_n$, CS

Note. $^+$ denotes positive excess kurtosis, $^-$ denotes negative excess kurtosis.
Source: author's work.

8. Conclusions

The main aim of the paper was to calculate GoFT powers for alternatives with undefined or constant γ_1 and $\bar{\gamma}_2$. 20 (symmetric alternatives) and 21 (asymmetric alternatives) recently recommended GoFTs were used in the Monte Carlo simulations for $n = 25$ or $n = 50$ and $\alpha = 0.05$. All tests were implemented in the R software (see Table 6 and the Appendix). The family of alternatives consisted of 24 distributions divided into three groups. The SM of a given alternative distribution to the normal distribution as a function of the scale parameter was defined. The scale parameter values were chosen to obtain appropriate SM values (e.g. equal 0.75, 0.50, 0.25), including the maximum SM value. If the SM equals 1, then the normal distribution is a special case of the alternative distribution.

The GoFT powers naturally increase when the sample size increases. It is noteworthy, however, that the GoFT power for the analysed alternatives and for a given sample size remain relatively unchanged despite the significant decrease in the SM. In short, the paper shows that many GoFTs proved unable to make a distinction between normal distribution and the alternatives. This situation occurs when the SM equals 0.25 or even 0.15.

The best tests for group I are the SJ and RJB tests ($\bar{\gamma}_2 > 0$) as well as the β_3^2 and CS tests ($\bar{\gamma}_2 < 0$), for group II the SJ, H1 and RJB tests, and for group III the ZA and CS tests. The simulation study also indicates that the AP, JB, AJB, SJ and RJB tests are not recommended for alternatives with $\bar{\gamma}_2 < 0$.

In this paper, H_0 states: data come from a normal distribution and H_1: data come from an other-than-normal distribution. Preliminary Monte Carlo studies have demonstrated that the same conclusions apply to the GoFTs with H_0: data come from the EXP(1) distribution (H_1: data come from an-other-than-EXP(1) distribution) or with H_0: data come from the LOG(1) distribution (H_1: data come from an-other-than-LOG(1) distribution) when the alternatives have undefined or constant $\gamma_1, \bar{\gamma}_2$.

Appendix

Figure. Different distributions and their similarity measure M to ND

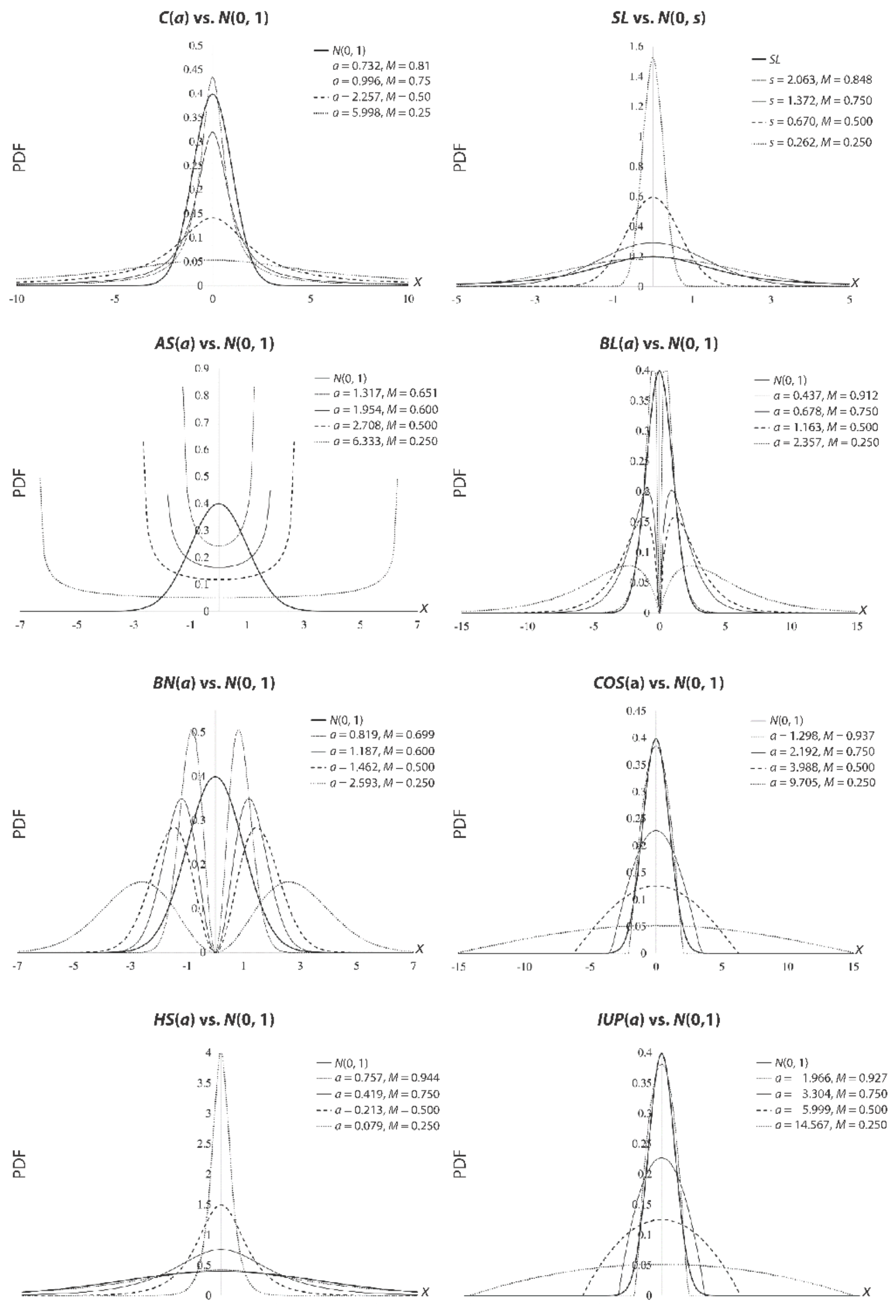


Figure. Different distributions and their similarity measure M to ND (cont.)

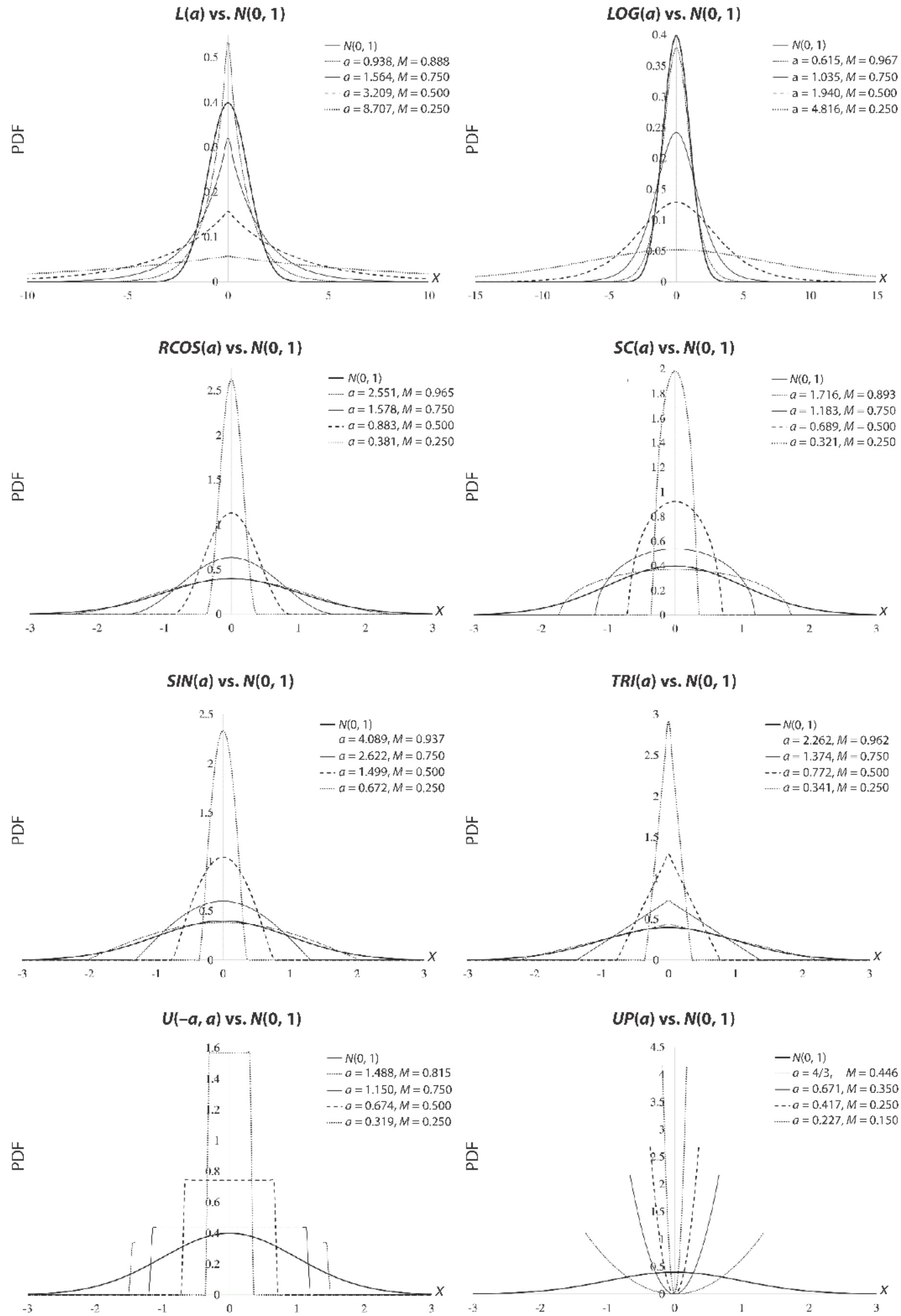
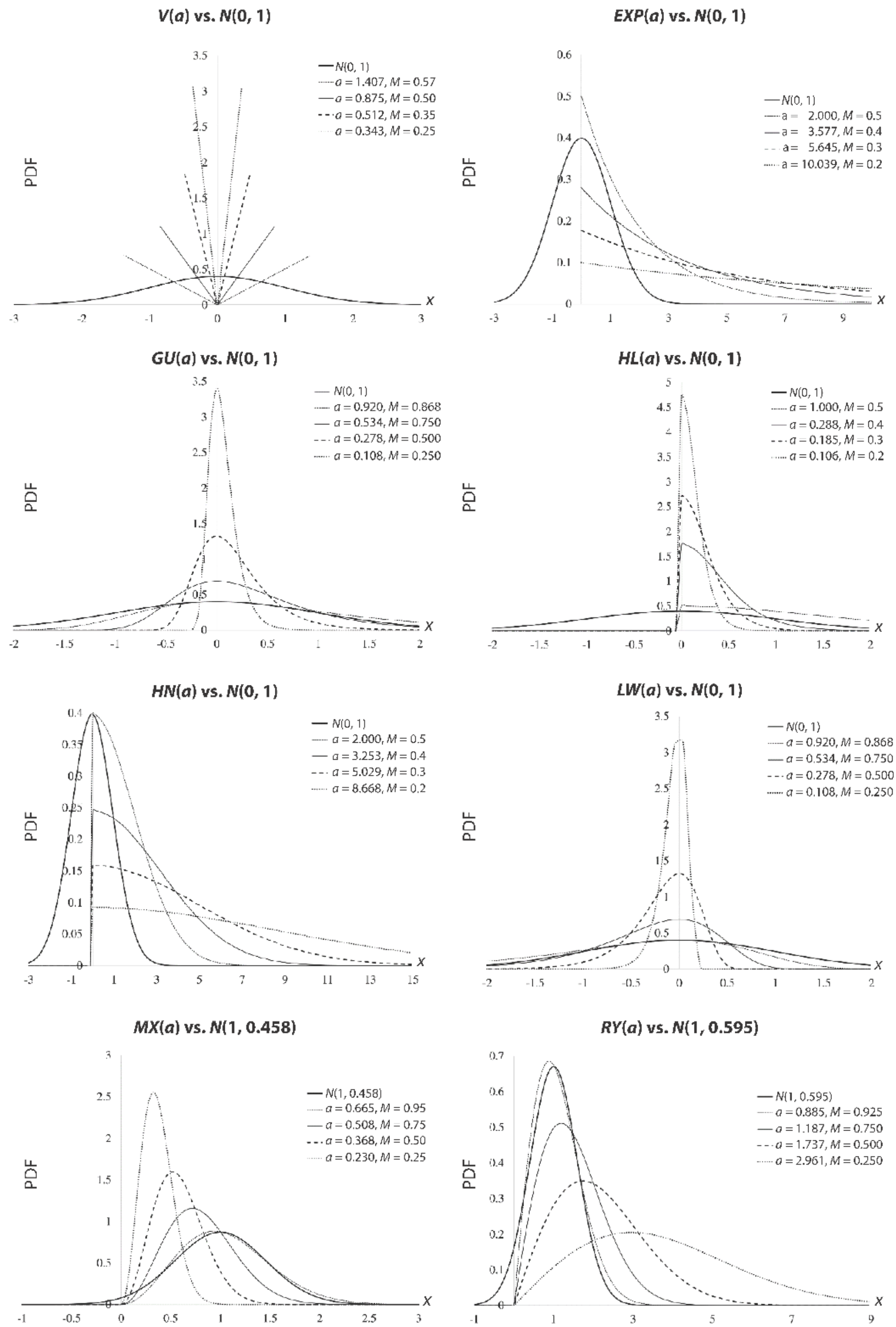


Figure. Different distributions and their similarity measure M to ND (cont.)



Source: author's work.

R codes

```
h=function(x) ((x-1)/(x+1))^2
Hn=function(x) {
x=sort((x-mean(x))/sd(x))
n=length(x)
Fn=1+1:n/n
F1=pnorm(x,0,1)+1
return(mean(h(F1/Fn))) }
Fn=function(i,n,a,b) ((i - a)/(n - a - b + 1))
LF=function(x,alfa,beta){
x=sort(x); n=length(x)
F=pnorm(x, mean(x), sd(x))
return(max(abs(Fn(1:n,n,alfa,beta)-F))) }
RJ=function(x){
x=sort(x); n=length(x)
z=qnorm(Fn1(1:n,n,3/8,3/8),0,1); s1=sum(x*z); s2=sum(z*z)
return(s1/sqrt(s2*(n-1)*var(x))) }
W1=function(u) qnorm(u)^2-1
T1n=function(x){
x=sort(x); n=length(x)
if (n==25) A1=-0.2114 else A1=-0.1297
if (n==25) B1=0.2323 else B1=0.34
s=sd(x)*sqrt((n-1)/n)
Fn=1:n/(n+1)
Cn=sum((W1(Fn)-A1)*x)/sqrt(n)
return(Cn^2/s^2/B1) }
TestSigma=function(x){
x=sort(x); Ft=pnorm(x,mean(x),sd(x))
n=length(x); Fn=1:n/n
licz=sum((abs(Ft-Fn))); mian=0
for (i in 1:n) {
mian=mian+max(Ft[i],Fn[i]) }
return(licz/mian) }
Bv=function(x){
x=sort(x); n=length(x)
mi=mean(x); sdev=sd(x)*sqrt((n-1)/n)
if (n==25) m=5 else m=15; s=0
for (i in 1:n){
up=i-m; if (up<1) up=1
uk=i+m; if(uk>n) uk=n
a=2*m/(x[uk]-x[up])/n
b=exp(-0.5*((x[i]-mi)/sdev)^2)/sdev/sqrt(2*pi)
s=s+((a-b)/(a+b))^2 }
return(s/n) }
```

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